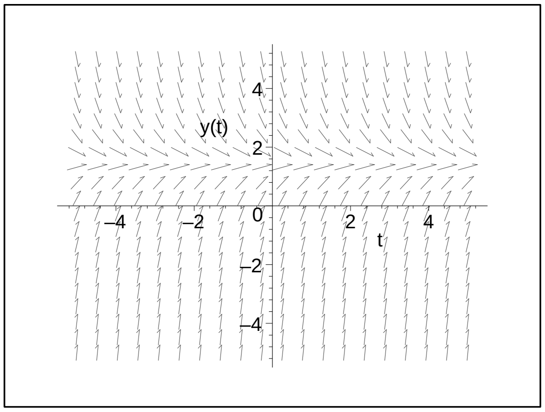
Chapter 1

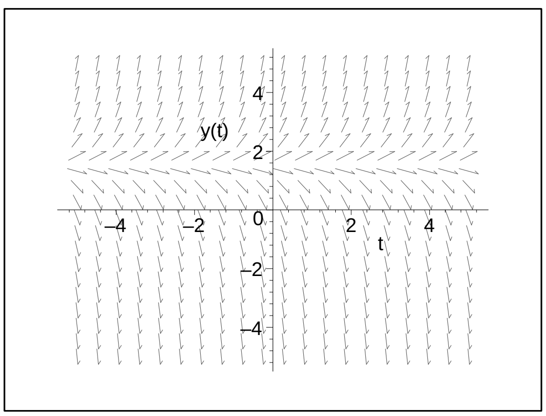
Section 1.1

1.



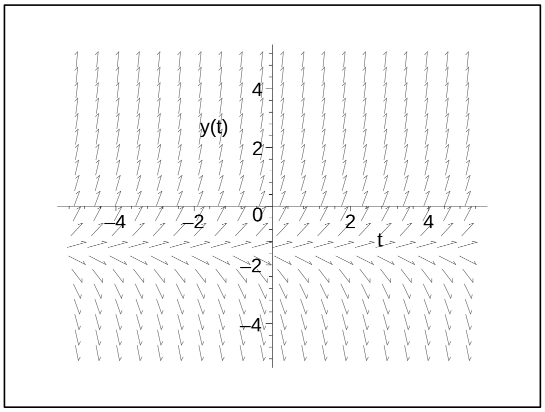
For $y > 3/2$, the slopes are negative, and, therefore the solutions decrease. For $y < 3/2$, the slopes are positive, and, therefore, the solutions increase. As a result, $y \to 3/2$ as $t \to \infty$

2.



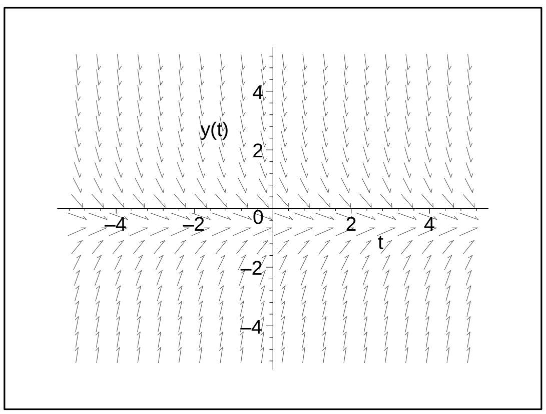
For $y > 3/2$, the slopes are positive, and, therefore the solutions increase. For $y < 3/2$, the slopes are negative, and, therefore, the solutions decrease. As a result, y diverges from $3/2$ as $t \to \infty$

3.



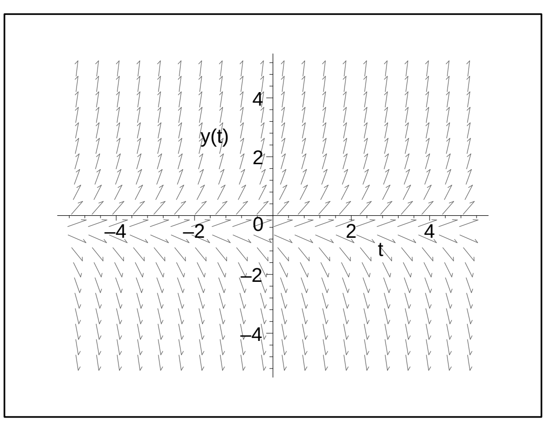
For $y > -3/2$, the slopes are positive, and, therefore the solutions increase. For $y < -3/2$, the slopes are negative, and, therefore, the solutions decrease. As a result, y diverges from $-3/2$ as $t \to \infty$

4.



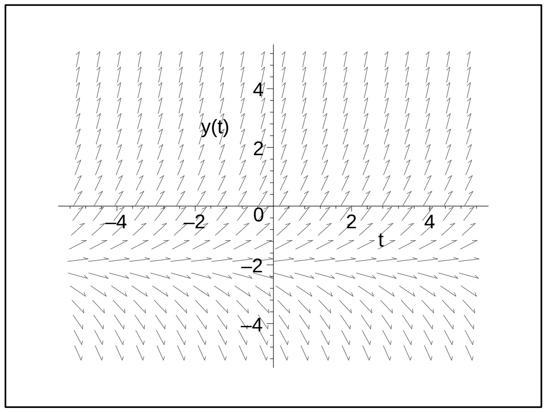
For $y > -1/2$, the slopes are negative, and, therefore the solutions decrease. For $y < -1/2$, the slopes are positive, and, therefore, the solutions increase. As a result, $y \to -1/2$ as $t \to \infty$

5.



For $y > -1/2$, the slopes are positive, and, therefore the solutions increase. For $y < -1/2$, the slopes are negative, and, therefore, the solutions decrease. As a result, y diverges from $-1/2$ as $t \to \infty$

6.



For $y > -2$, the slopes are positive, and, therefore the solutions increase. For $y < -2$, the slopes are negative, and, therefore, the solutions decrease. As a result, y diverges from -2 as $t \to \infty$

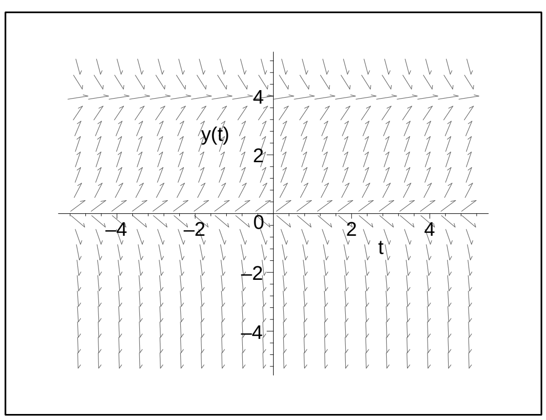
7. For the solutions to satisfy $y \to 3$ as $t \to \infty$, we need $y' < 0$ for $y > 3$ and $y' > 0$ for $y < 3$. The equation $y' = 3 - y$ satisfies these conditions.

8. For the solutions to satisfy $y \to 2/3$ as $t \to \infty$, we need $y' < 0$ for $y > 2/3$ and $y' > 0$ for $y < 2/3$. The equation $y' = 2 - 3y$ satisfies these conditions.

9. For the solutions to satisfy y diverges from 2, we need $y' > 0$ for $y > 2$ and $y' < 0$ for $y < 2$. The equation $y' = y - 2$ satisfies these conditions.

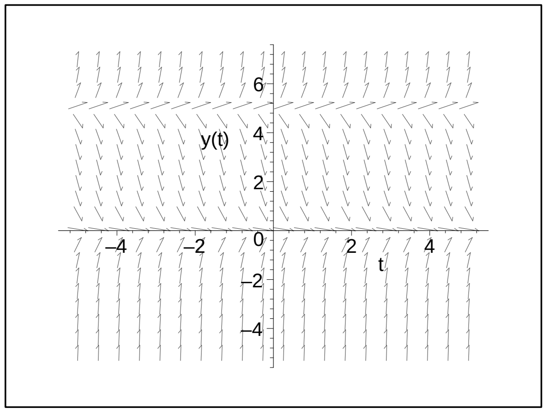
10. For the solutions to satisfy y diverges from $1/3$, we need $y' > 0$ for $y > 1/3$ and $y' < 0$ for $y < 1/3$. The equation $y' = 3y - 1$ satisfies these conditions.

11.



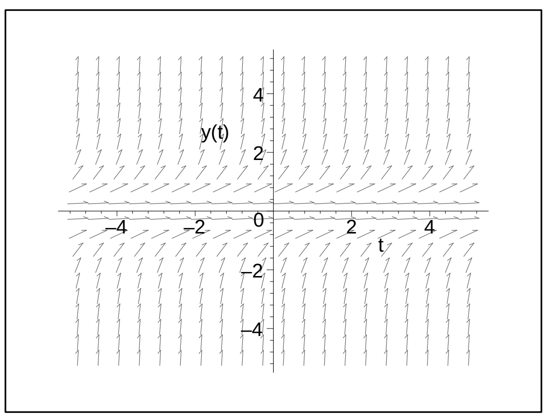
$y = 0$ and $y = 4$ are equilibrium solutions; $y \to 4$ if initial value is positive; y diverges from 0 if initial value is negative.

12.



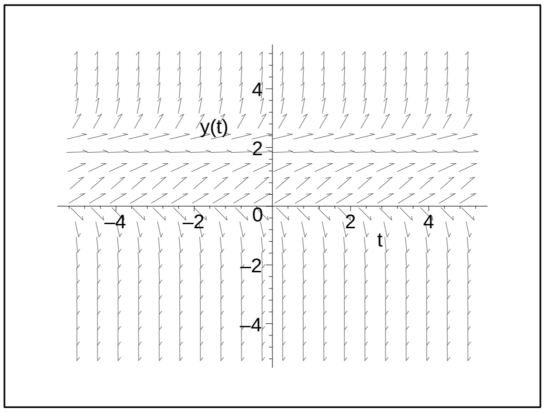
$y = 0$ and $y = 5$ are equilibrium solutions; y diverges from 5 if initial value is greater than 5; $y \to 0$ if initial value is less than 5.

13.



$y = 0$ is equilibrium solution; $y \to 0$ if initial value is negative; y diverges from 0 if initial value is positive.

14.



$y = 0$ and $y = 2$ are equilibrium solutions; y diverges from 0 if initial value is negative; $y \to 2$ if initial value is between 0 and 2; y diverges from 2 if initial value is greater than 2.

15. (j)

16. (c)

17. (g)

18. (b)

19. (h)

20. (e)

21.

(a) Let $q(t)$ denote the amount of chemical in the pond at time t. The chemical q will be measured in grams and the time t will be measured in hours. The rate at which the chemical is entering the pond is given by 300 gallons/hour $\cdot$.01 grams/gal $= 300 \cdot 10^{-2}$. The rate at which the chemical leaves the pond is given by 300 gallons/hour $\cdot q/1,000,000$ grams/gal $= 300 \cdot q10^{-6}$. Therefore, the differential equation is given by $dq/dt = 300(10^{-2} - q10^{-6})$.

(b) As $t \to \infty$, $10^{-2} - q10^{-6} \to 0$. Therefore, $q \to 10^4$ g. The limiting amount does not depend on the amount that was present initially.

22. The surface area of a spherical raindrop of radius r is given by $S = 4\pi r^2$. The volume of a spherical raindrop is given by $V = 4\pi r^3/3$. Therefore, we see that the surface area $S = cV^{2/3}$ for some constant c. If the raindrop evaporates at a rate proportional to its surface area, then $dV/dt = -kV^{2/3}$ for some $k > 0$.

23. The difference between the temperature of the object and the ambient temperature is $u - 70$. Since the difference is decreasing if $u > 70$ (and increasing if $u < 70$) and the rate constant is 0.05, the corresponding differential equation is given by $du/dt = -0.05(u - 70)$ where u is measured in degrees Fahrenheit and t is measured in minutes.

24.

(a) Let $q(t)$ be the total amount of the drug (in milligrams) in the body at a given time t (measured in hours). The drug enters the body at the rate of 5 mg/cm^3 $\cdot$100 cm^3/hr

= 500 mg/hr, and the drug leaves the body at the rate of $0.4q$ mg/hr. Therefore, the governing differential equation si given by $dq/dt = 500 - 0.4q$.

(b) If $q > 1250$, then $q' > 0$. If $q < 1250$, then $q' > 0$. Therefore, $q \to 1250$.

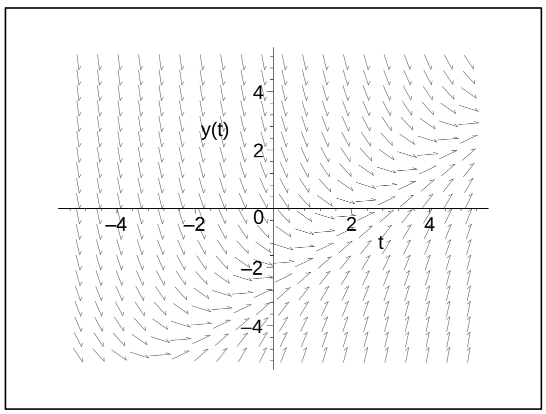
25.

(a) Following the discussion in the text, the equation is given by $mv' = mg - kv^2$.

(b) After a long time, $v' \to 0$. Therefore, $mg - kv^2 \to 0$, or $v \to \sqrt{mg/k}$

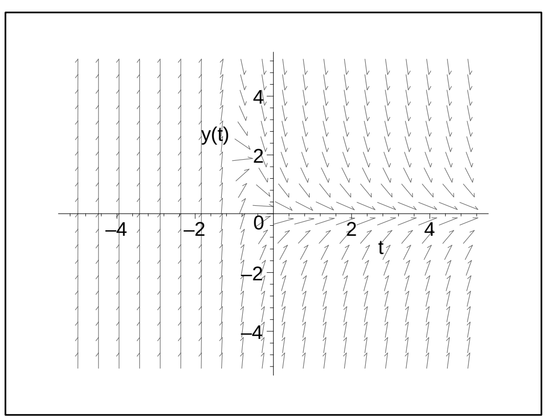
(c) We need to solve the equation $\sqrt{.025 \cdot 9.8/k} = 35$. Solving this equation, we see that $k = 0.0002$ kg/m

26.



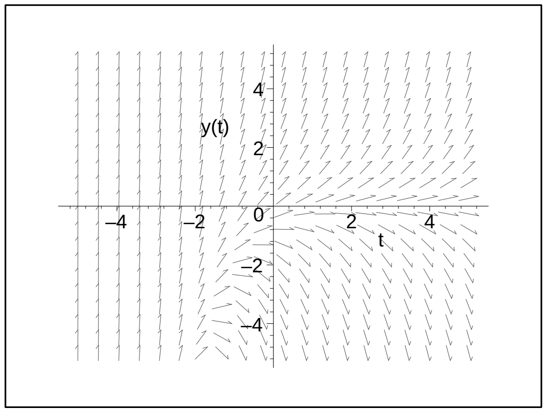
y is asymptotic to $t - 3$ as $t \to \infty$

27.



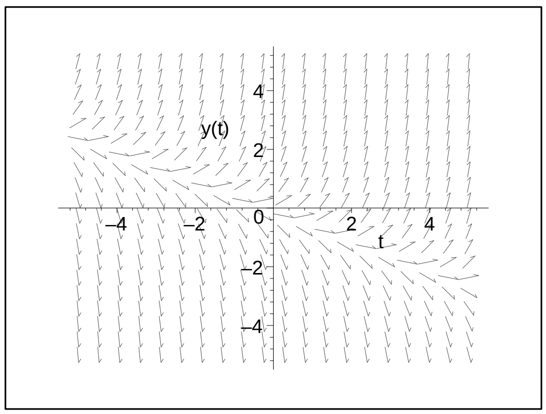
$y \to 0$ as $t \to \infty$

28.



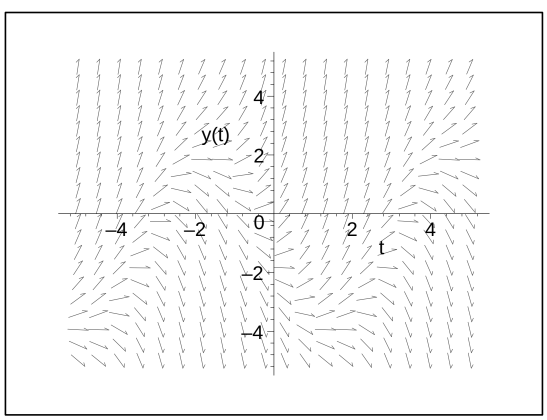
$y \to \infty, 0$, or $-\infty$ depending on the initial value of y

29.



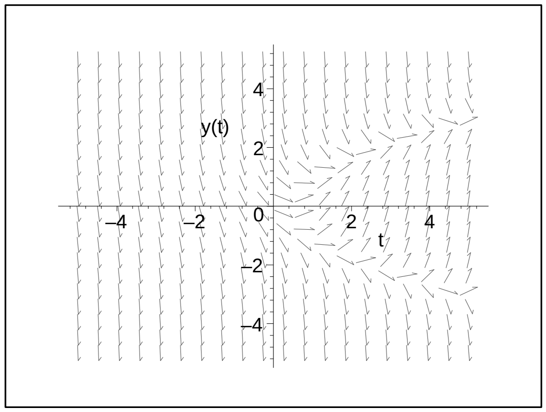
$y \to \infty$ or $-\infty$ depending whether the initial value lies above or below the line $y = -t/2$.

30.



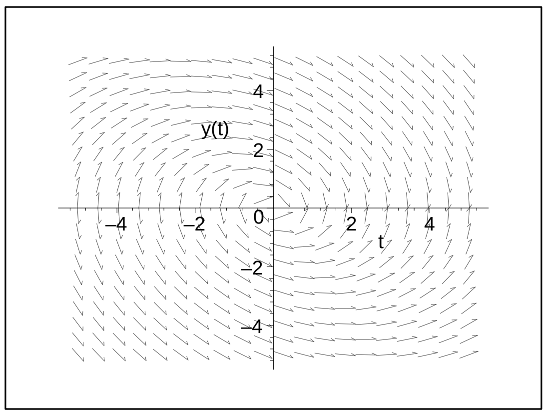
$y \to \infty$ or $-\infty$ or y oscillates depending whether the initial value of y lies above or below the sinusoidal curve.

31.



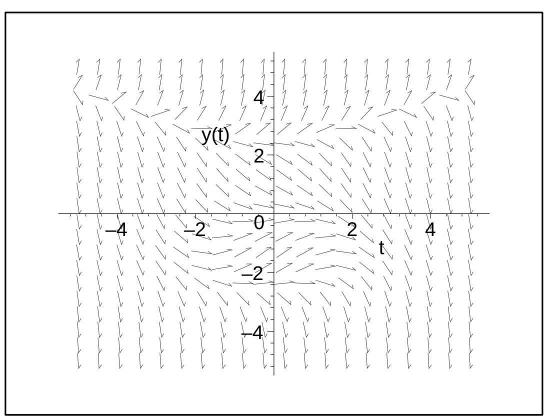
$y \to -\infty$ or is asymptotic to $\sqrt{2t-1}$ depending on the initial value of y

32.



$y \to 0$ and then fails to exist after some $t_f \geq 0$

33.



$y \to \infty$ or $-\infty$ depending on the initial value of y

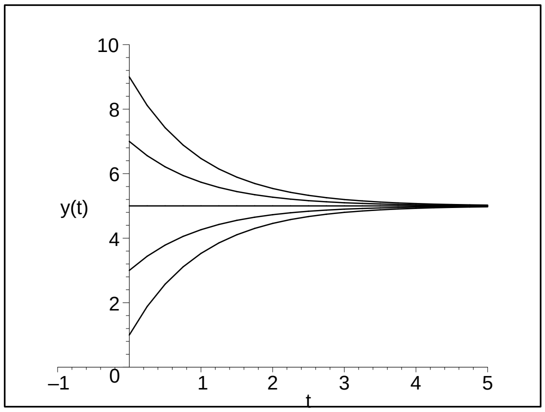
Section 1.2

1.

(a) Rewrite the equation as

$$\frac{dy}{5-y} = dt$$

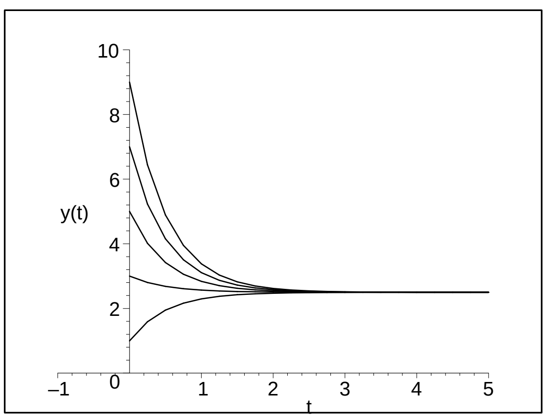
and then integrate both sides. Doing so, we see that $-\ln|5-y| = t + c$. Applying the exponential function, we have $5 - y = ce^{-t}$. Substituting in our initial condition $y(0) = y_0$, we have $5 - y_0 = c$. Therefore, our solution is $y(t) = 5 + (y_0 - 5)e^{-t}$.



(b) Rewrite the equation as

$$\frac{dy}{5-2y} = dt$$

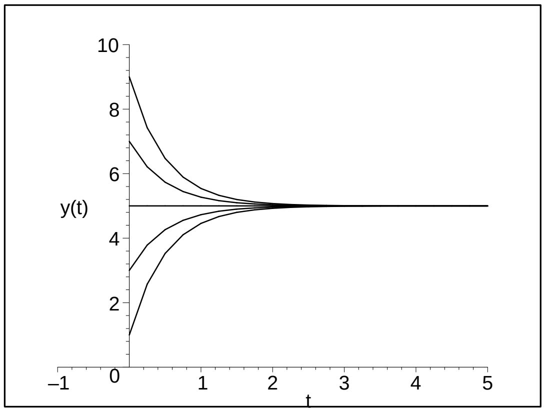
and then integrate both sides. Doing so, we see that $\ln|5-2y| = -2t + c$. Applying the exponential function, we have $5 - 2y = ce^{-2t}$. Substituting in our initial condition $y(0) = y_0$, we have $5 - 2y_0 = c$. Therefore, our solution is $y(t) = (5/2) + [y_0 - (5/2)]e^{-2t}$



(c) Rewrite the equation as

$$\frac{dy}{10-2y} = dt$$

and then integrate both sides. Doing so, we see that $\ln|10-2y| = -2t + c$. Applying the exponential function, we have $10 - 2y = ce^{-2t}$. Substituting in our initial condition $y(0) = y_0$, we have $10 - 2y_0 = c$. Therefore, our solution is $y(t) = 5 + [y_0 - 5]e^{-2t}$



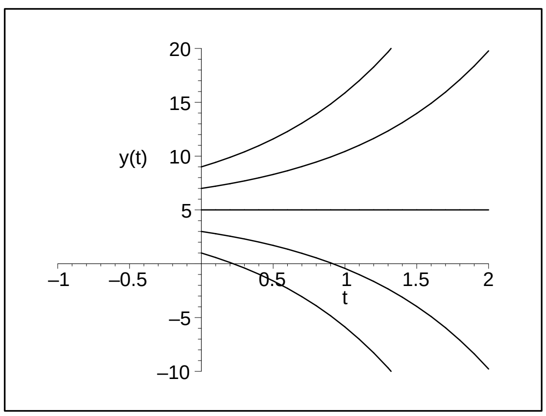
The equilibrium solution is $y = 5$ in (a) and (c), $y = 5/2$ in (b). The solution approaches equilibrium faster in (b) and (c) than in (a).

2.

(a) Rewrite the equation as

$$\frac{dy}{y-5} = dt$$

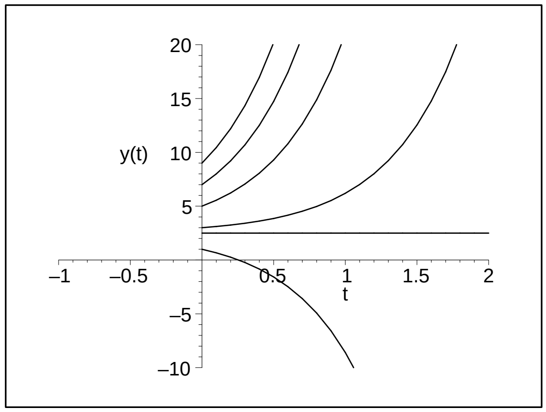
and then integrate both sides. Doing so, we see that $\ln|y-5| = t + c$. Applying the exponential function, we have $y - 5 = ce^t$. Substituting in our initial condition $y(0) = y_0$, we have $y_0 - 5 = c$. Therefore, our solution is $y(t) = 5 + [y_0 - 5]e^t$



(b) Rewrite the equation as

$$\frac{dy}{2y-5} = dt$$

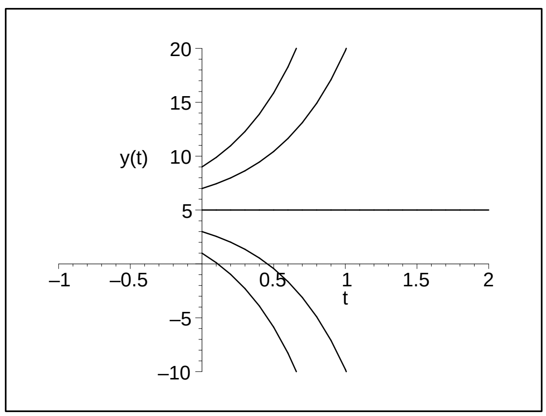
and then integrate both sides. Doing so, we see that $\ln|2y-5| = 2t + c$. Applying the exponential function, we have $2y - 5 = ce^{2t}$. Substituting in our initial condition $y(0) = y_0$, we have $2y_0 - 5 = c$. Therefore, our solution is $y(t) = (5/2) + [y_0 - (5/2)]e^{2t}$



(c) Rewrite the equation as

$$\frac{dy}{2y-10} = dt$$

and then integrate both sides. Doing so, we see that $\ln|2y-10| = 2t + c$. Applying the exponential function, we have $2y - 10 = ce^{2t}$. Substituting in our initial condition $y(0) = y_0$, we have $2y_0 - 10 = c$. Therefore, our solution is $y(t) = 5 + [y_0 - 5]e^{2t}$



The equilibrium solution is $y = 5$ in (a) and (c), $y = 5/2$ in (b); solution diverges from equilibrium faster in (b) and (c) than in (a).

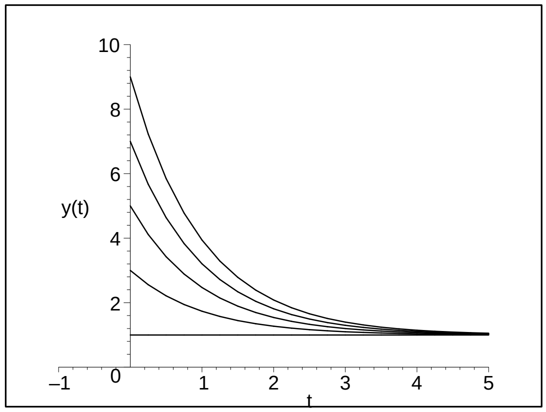
3.

(a) Rewrite the equation as

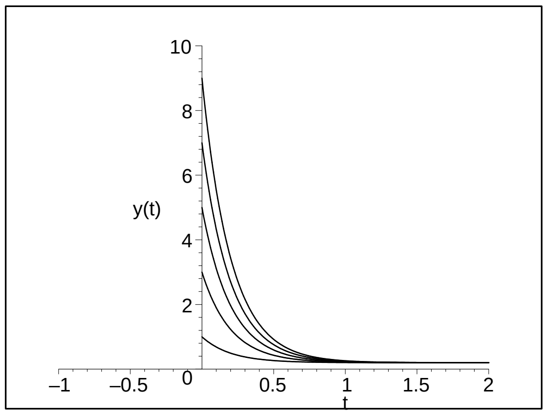
$$\frac{dy}{b-ay} = dt$$

and then integrate both sides. Doing so, we see that $\ln|b-ay| = -at + c$. Applying the exponential function, we have $b - ay = ce^{-at}$, or $y = ce^{-at} + (b/a)$

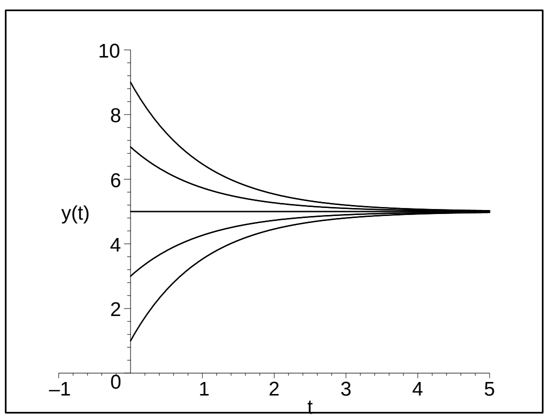
(b) Below we show solution curves for various initial conditions under the cases $a = 1, b = 1$, $a = 5, b = 1$, $a = 1, b = 5$ and $a = 5, b = 5$, respectively.



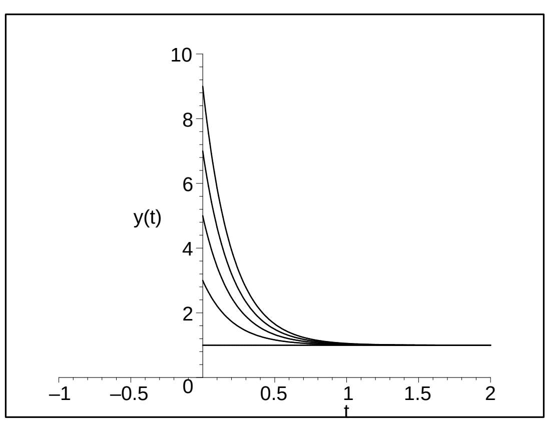
10
8
6
y(t)
4
2
0
–1
1
2
3
4
5
t



10
8
6
y(t)
4
2
0
–1
–0.5
0.5
1
1.5
2
t



10
8
6
y(t)
4
2
0
–1
1
2
3
4
5
t



10
8
6
y(t)
4
2
0
–1
–0.5
0.5
1
1.5
2
t

(c) (i) As a increases, the equilibrium is lower and is approached more rapidly. (ii) As b increases, the equilibrium is higher. (iii) As a and b increase, but a/b remains the same, the equilibrium remains the same and is approached more rapidly.

4.

(a) The equilibrium solution occurs when $dy/dt = ay - b = 0$. Therefore, the equilibrium solution is $y_e = b/a$

(b) If $Y(t) = y - y_e$, then $Y'(t) = y' - y_e' = y' = ay - b = a(Y + y_e) - b = aY + ay_e - b = aY + a(b/a) - b = 0 = aY$. Therefore, Y satisfies the equation $Y' = aY$.

5. The solution of the homogeneous problem is $y = ce^{-2t}$. Therefore, we assume the solution will have the form $y = ce^{-2t} + At + B$. Substituting a function of this form into the differential equation leads to the equation

$$2At + A + 2B = t - 3.$$

Equating like coefficients, we see that $A = 1/2$ and $B = -7/4$. Therefore, the general solution is

$$y = ce^{-2t} + \frac{1}{2}t - \frac{7}{4}.$$

6. The solution of the homogeneous problem is $y = ce^{3t}$. Therefore, we assume the solution will have the form $y = ce^{3t} + Ae^{-t}$. Substituting a function of this form into the differential equation leads to the equation

$$-4Ae^{-t} = e^{-t}.$$

Equating like coefficients, we see that $A = -1/4$. Therefore, the general solution is

$$y = ce^{3t} - \frac{1}{4}e^{-t}.$$

7. The solution of the homogeneous problem is $y = ce^{-t}$. Therefore, we assume the solution will have the form $y = ce^{-t} + A\cos(2t) + B\sin(2t)$. Substituting a function of this form into the differential equation leads to the equation

$$[-2A + B]\sin(2t) + [2B + A]\cos(2t) = 3\cos(2t).$$

Solving the two equations, $-2A + B = 0$ and $2B + A = 3$, we see that $A = 3/5$ and $B = 6/5$. Therefore, the general solution is

$$y = ce^{-t} + \frac{3}{5}\cos(2t) + \frac{6}{5}\sin(2t).$$

8. The solution of the homogeneous problem is $y = ce^{2t}$. Therefore, we assume the solution will have the form $y = ce^{2t} + A\cos(t) + B\sin(t)$. Substituting a function of this form into the differential equation leads to the equation

$$[-A - 2B]\sin(t) + [B - 2A]\cos(t) = 2\sin(t).$$

Solving the system of equations $-A - 2B = 2$ and $B - 2A = 0$, we see that $A = -2/5$ and $B = -4/5$. Therefore, the general solution is

$$y = ce^{2t} - \frac{2}{5}\cos(t) - \frac{4}{5}\sin(t).$$

9. The solution of the homogeneous problem is $y = ce^{-2t}$. Therefore, we assume the solution will have the form $y = ce^{-2t} + At + B + C\cos(t) - D\sin(t)$. Substituting a function of this form into the differential equation leads to the equation

$$2At + [A + 2B] + [C + 2D]\cos(t) + [2C - D]\sin(t) = 2t + 3\sin(t).$$

Equating like coefficients, we see that $A = 1$, $B = -1/2$, $C = 6/5$ and $D = -3/5$. Therefore, the general solution is

$$y = ce^{-2t} + t - \frac{1}{2} + \frac{6}{5}\sin(t) - \frac{3}{5}\cos(t).$$

10. The solution of the homogeneous problem is $y = ce^{2t}$. Therefore, we assume the solution will have the form $y = ce^{2t} + Ae^t + Bt^2 + Ct + D$. Substituting a function of this form into the differential equation leads to the equation

$$-Ae^t - 2Bt^2 + [2B - 2C]t + [C - 2D] = 3e^t + t^2 + 1.$$

Equating like coefficients, we see that $A = -3$, $B = -1/2$, $C = -1/2$ and $D = -3/4$. Therefore, the general solution is

$$y = ce^{2t} - 3e^t - \frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}.$$

11.

(a) The general solution is $p(t) = 900 + ce^{t/2}$. Plugging in for the initial condition, we have $p(t) = 900 + (p_0 - 900)e^{t/2}$. With $p_0 = 850$, the solution is $p(t) = 900 - 50e^{t/2}$. To find the time when the population becomes extinct, we need to find the time T when $p(T) = 0$. Therefore, $900 = 50e^{T/2}$, which implies $e^{T/2} = 18$, and, therefore, $T = 2\ln 18 \cong 5.78$ months.

(b) Using the general solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, we see that the population will become extinct at the time T when $900 = (900 - p_0)e^{T/2}$. That is, $T = 2\ln[900/(900 - p_0)]$ months

(c) Using the general solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, we see that the population after 1 year (12 months) will be $p(6) = 900 + (p_0 - 900)e^6$. If we want to know the initial population which will lead to extinction after 1 year, we set $p(6) = 0$ and solve for p_0. Doing so, we have $(900 - p_0)e^6 = 900$ which implies $p_0 = 900(1 - e^{-6}) \cong 897.8$

12.

(a) The general solution is $p(t) = p_0 e^{rt}$, where t is measured in days. If the population doubles in 30 days, then $p(30) = 2p_0 = p_0 e^{30r}$. Therefore, $r = (\ln 2)/30$ day^{-1}.

(b) If the population doubles in N days, then $p(N) = 2p_0 = p_0 e^{Nr}$. Therefore, $r = (\ln 2)/N$ day^{-1}

13.

(a) The solution is given by $v(t) = 35(1 - e^{-0.28t})$. The limiting velocity is 35 m/sec. Therefore, we want to find the time T when $v(T) = .98 \cdot 35 = 34.3$ m/sec. Plugging this value into our equation for v, we have $34.3 = 35(1 - e^{-0.28T})$, or $e^{-0.28T} = .02$ which implies $T = (\ln 50)/0.28 \cong 13.97$ sec

(b) To find the position, we integrate the velocity function above. For $v(t) = 35(1 - e^{-0.28t})$, the height is given by $s(t) = \int v(t) = 35t + 125te^{-0.28t}\,dt + C$. Assuming, $s(0) = 0$, we see that $c = -125$. Therefore, $s(t) = 35t + 125e^{-0.28t} - 125$. When $T = 13.97$ seconds, we see that the distance traveled is approximately 366.5 m.

14.

(a) Assuming no air resistance, Newton's Second Law can be expressed as

$$m\frac{dv}{dt} = mg$$

where g is the gravitational constant. Dividing the above equation by m and assuming that the initial velocity is zero, we see that our initial value problem is $dv/dt = 9.8$, $v(0) = 0$

(b) We are assuming the object is released from a height of 300 meters above the ground. The height at a later time t satisfies $ds/dt = v = 9.8t$. Taking the point of release as the origin and integrating the above equation for s, we have $s(t) = 4.9t^2$. We need to find the time T when $s(T) = 300$. That is, $4.9T^2 = 300$. Solving this equation, we have $T = \sqrt{300/4.9} \cong 7.82$ sec

(c) Using the equation $v = 9.8t$, we see that when $T \cong 7.82$ seconds, $v \cong 76.68$ m/sec

15.

(a) If we are assuming that the drag force is proportional to the square of the velocity, equation (1) becomes

$$m\frac{dv}{dt} = mg - \gamma v^2.$$

Plugging in $m = 0.025$, $g = 9.8$, the equation can be written as

$$\frac{dv}{dt} = 9.8 - \frac{\gamma}{.025}v^2.$$

If the limiting velocity is 35 m/sec, then $\gamma(35)^2 = 9.8 \cdot .025$ which implies $\gamma = 0.0002$. Therefore,

$$\frac{dv}{dt} = 9.8 - 0.008v^2,$$

or

$$\frac{dv}{dt} = [(35)^2 - v^2]/125.$$

(b) The equation can be rewritten as

$$\frac{dv}{(35)^2 - v^2} = \frac{dt}{125}.$$

Integrating both sides, we have

$$\ln\left|\frac{v+35}{v-35}\right| = \frac{70}{125}t + c.$$

Plugging in the initial condition $v(0) = 0$, we have $c = 0$. Applying the exponential function to both sides of the equation, we have

$$v + 35 = e^{70t/125}(35 - v).$$

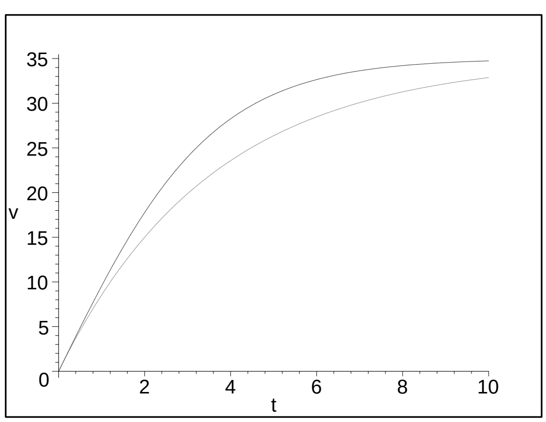
Solving this equation for v, we have

$$v(t) = 35\left[\frac{e^{70t/125} - 1}{e^{70t/125} + 1}\right]$$

or

$$v(t) = 35\left[\frac{e^{35t/125}(e^{35t/125} - e^{-35t/125})}{e^{35t/125}(e^{35t/125} + e^{-35t/125})}\right] = 35\tanh(7t/25)$$

(c) Below we show the graphs of $v(t)$ above (the top curve) and the solution to the problem in example 2 (the bottom curve)



(d) The quadratic force leads to the falling object attaining its limiting velocity sooner.

(e) The distance $x(t) = \int v(t)\,dt = \int 35\tanh(7t/25)\,dt = 125\ln\cosh(7t/25)$.

(f) Plugging 300 in for $x(T)$ in the answer to part (d), we have $300 = 125\ln\cosh(7T/25)$. Therefore, $T = (25/7)\text{arccosh}(e^{12/5}) \cong 11.04$ sec

16.

(a) The general solution of the equation is $Q(t) = ce^{-rt}$. Given that $Q(0) = 100$, we have $c = 100$. Assuming that $Q(1) = 82.04$, we have $82.04 = 100e^{-r}$. Solving this equation for r, we have $r = -\ln(82.04/100) = .19796$ per week or $r = 0.02828$ per day.

(b) Using the form of the general solution and r found above, we have $Q(t) = 100e^{-0.02828t}$

(c) Let T be the time it takes the isotope to decay to half of its original amount. From part (b), we conclude that $.5 = e^{-0.2828T}$ which implies that $T = -\ln(0.5)/0.2828 \cong 24.5$ days

17. The general solution of the differential equation is $Q(t) = Q_0e^{-rt}$ where $Q_0 = Q(0)$. Let τ be the half-life. Plugging τ into the equation for Q, we have $0.5Q_0 = Q_0e^{-r\tau}$. Therefore, $0.5 = e^{-r\tau}$ which implies $\tau = -\ln(0.5)/r = \ln(2)/r$. Therefore, we conclude that $r\tau = \ln 2$.

18. The differential equation for radium-226 is $dQ/dt = -rQ$. The solution of this equation is $Q(t) = Q_0e^{-rt}$. Using the result from exercise 17 and the fact that the half-life is 1620 years, we conclude that the decay rate $r = \ln(2)/\tau = \ln(2)/1620$. Let T be the time it takes for the isotope to decay to 3/4 of its original amount. Then

$$\frac{3}{4}Q_0 = Q_0e^{-\ln(2)T/1620}$$

which implies $T = -1620\ln(3/4)/\ln(2) \cong 672.4$ years.

19.

(a) We rewrite the equation as

$$\frac{du}{u-T} = -k.$$

Integrating both sides, we have $\ln|u-T| = -kt + c$. Applying the exponential function to both sides of the equation and plugging in the initial condition $u(0) = u_0$, we arrive at the general solution $u(t) = T + (u_0 - T)e^{-kt}$

(b) Since T is a constant, we see that if u satisfies the equation $du/dt = -k(u-T)$, then $d(u-T)/dt = du/dt = -k(u-T)$. Then using the result from exercise 17 above, we know that the relationship between the decay rate k and the time τ when the temperature difference is reduced by half satisfies the relationship $k\tau = \ln 2$.

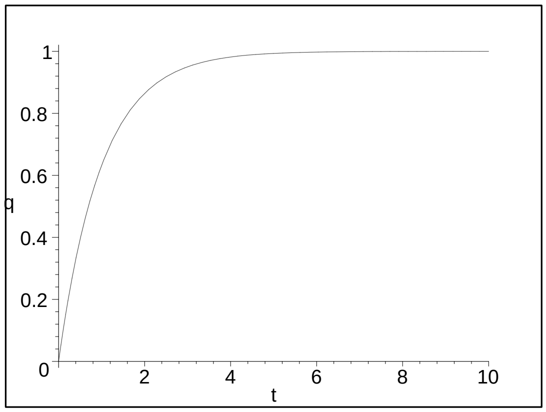
20. Based on exercise 19 above, the differential equation for the temperature in the room is given by

$$\frac{du}{dt} = -.15(u-10)$$

with an initial condition of $u(0) = 70$. As shown in exercise 19 above, the solution is given by $u(t) = 10 + 60e^{-0.15t}$. We need to find the time t such that $u(t) = 32$. That is, $22 = 60e^{-0.15t}$. Solving this equation for t, we have $t = -\ln(22/60)/0.15 \cong 6.69$ hours.

21.

(a) The solution of the differential equation with $q(0) = 0$ is $q(t) = CV(1 - e^{-t/RC})$. Below we show a sketch in the case when $C = V = R = 1$.

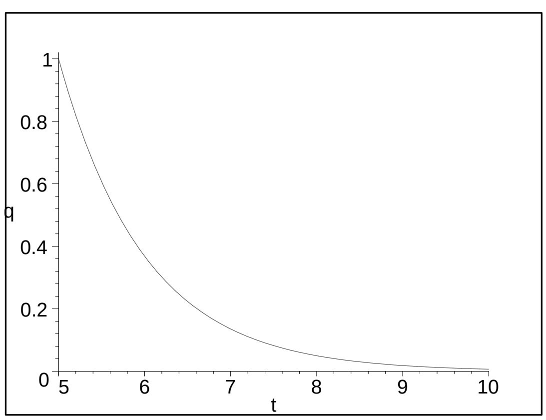


(b) As $t \to \infty$, the exponential term vanishes. Therefore, the limiting value is $q_L = CV$

(c) If the battery is removed, then $V = 0$. Therefore, our differential equation is

$$R\frac{dq}{dt} + \frac{q}{C} = 0.$$

Also, we are assuming that $q(t_1) = q_L = CV$. Solving the differential equation, we have $q = ce^{-t/RC}$. Using the initial condition $q(t_1) = CV$, we have $q(t_1) = ce^{-t_1/RC} = CV$. Therefore, $c = CVe^{t_1/RC}$. We conclude that $q(t) = CV\exp[-(t - t_1)/RC]$ Below we show a graph of the solution taking $C = V = R = 1$ and $t_1 = 5$.



22.

(a) The accumulation rate of the chemical is (0.01)(300) grams per hour. At any given time t, the concentration of the chemical in the pond is $Q(t)/10^6$ grams per gallon. Therefore, the chemical leaves the pond at the rate of $300Q(t)/10^6$ grams per hour. Therefore, the equation for Q is given by $Q' = 3(1 - 10^{-4}Q)$. Since initially there are no chemicals in the pond, $Q(0) = 0$.

(b) Rewrite the equation as

$$\frac{dQ}{10000 - Q} = 0.0003dt.$$

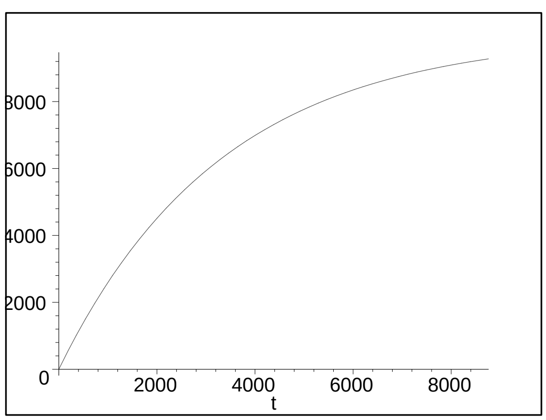
Integrating both sides of this equation, we have $\ln|10000 - Q| = -0.0003t + C$. Applying the exponential function to both sides of this equation, we have $10000 - Q = ce^{-0.0003t}$. Assuming $Q(0) = 0$, we see that $c = 10000$. Therefore, $Q(t) = 10000(1 - e^{-0.0003t})$ where t is measured in hours. Since 1 year is 8760 hours, we see that the amount of chemical in the pond after 1 year is $Q(8760) = 10000(1 - e^{-0.0003t}) \cong 9277.77$ grams.

(c) With the accumulation rate now equal to zero, the equation becomes $dQ/dt = -0.0003Q(t)$ grams/hour. Resetting the time variable, we assign the new initial value as $Q(0) = 9277.77$ grams.

(d) The solution of the differential equation is $Q(t) = 9277.77e^{-0.0003t}$ after t hrs. Therefore, after 1 year $Q(8760) \cong 670.07$ g

(e) Letting T be the amount of time after the source is removed, we obtain the equation $10 = 9277.77e^{-0.0003t}$. Solving this equation, we have $T - \ln(10/9277.77)/0.0003 \cong 2.60$ years

(f)



23.

(a) We are assuming that no dye is entering the pool. The rate at which the dye is leaving the pool is given by $200 \cdot (q/60,000)$ g/min $= q/300$ g/min. Since initially, there are 5 kg of the dye in the pool, the initial value problem is $q' = -q/300$, $q(0) = 5000$ g

(b) The solution of this initial value problem is $q(t) = 5000e^{-t/300}$ where g is in grams and t is in minutes.

(c) In 4 hours (240 minutes), the amount of dye in the pool will be $q(240) \cong 2246.6$ grams. Since there is 60,000 gallons of water in the pool, the concentration will be $2246.6/60,000 \cong 0.0374$ grams/gallon. So, no, the pool will not be reduced to the desired level within 4 hours.

(d) Let T be the time that it takes to reduce the concentration level of the dye to 0.02 grams/gallon. At that time, the amount of dye in the pool needs to be 1200 grams (as $1200/60000 = 0.02$). Plugging $q(T) = 1200$ into our equation for q, we have $1200 = 5000e^{-T/300}$. Solving this equation, we have $T = 300\ln(25/6) \cong 7.136$ hr

(e) Let r be the necessary flow rate. As in part (a), if the water leaves the pool at the rate of r gallons/minute, then the initial value problem will be $q' = -rq/60,000$, $q(0) = 5000$. The solution of this initial value problem is given by $q(t) = 5000e^{-rt/60,000}$. We need to find the decay rate r such that when $t = 240$ minutes, the amount of dye $q = 1200$ grams. That is, we need to solve the equation $1200 = 5000e^{-240r/60,000}$. Solving this equation, we have $r = 250\ln(25/6) \cong 256.78$ gal/min

Section 1.3

1. The Euler formula is given by $y_{n+1} = y_n + h(3 + t_n - y_n) = (1 - h)y_n + h(3 + t_n)$.

(a) 1.2, 1.39, 1.571, 1.7439

(b) 1.1975, 1.38549, 1.56491, 1.73658

(c) 1.19631, 1.38335, 1.56200, 1.73308

(d) The differential equation is linear with solution $y(t) = 2 + t - e^{-t}$.

1.19516, 1.38127, 1.55918, 1.72968

2. The Euler formula is given by $y_{n+1} = y_n + h(2y_n - 1) = (1 + 2h)y_n - h$.

(a) 1.1, 1.22, 1.364, 1.5368

(b) 1.105, 1.23205, 1.38578, 1.57179

(c) 1.10775, 1.23873, 1.39793, 1.59144

(d) The differential equation is linear with solution $y(t) = (1 + e^{2t})/2$.

1.1107, 1.24591, 1.41106, 1.61277

3. The Euler formula is given by $y_{n+1} = y_n + h(0.5 - t_n + 2y_n) = (1 + 2h)y_n + h(0.5 - t_n)$.

(a) 1.25, 1.54, 1.878, 2.2736

(b) 1.26, 1.5641, 1.92156, 2.34359

(c) 1.26551, 1.57746, 1.94586, 2.38287

(d) The differential equation is linear with solution $y(t) = 0.5t + e^{2t}$.

1.2714, 1.59182, 1.97212, 2.42554

4. The Euler formula is given by $y_{n+1} = y_n + h(3\cos(t_n) - 2y_n) = (1-2h)y_n + 3h\cos(t_n)$.

(a) 0.3, 0.538501, 0.724821, 0.866458

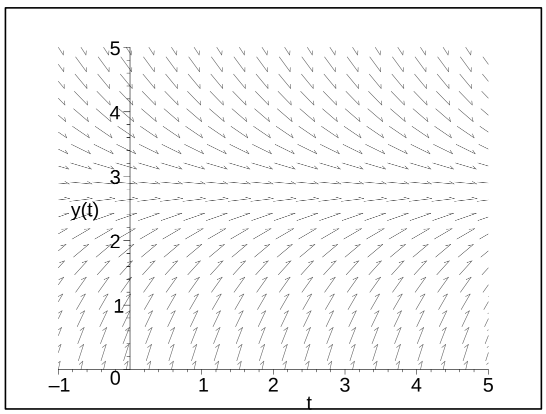
(b) 0.284813, 0.513339, 0.693451, 0.831571

(c) 0.277920, 0.501813, 0.678949, 0.815302

(d) The differential equation is linear with solution $y(t) = (6\cos(t) + 3\sin(t) - 6e^{-2t})/5$.

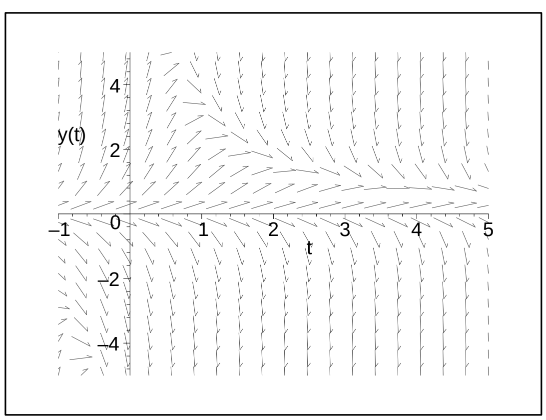
0.271428, 0.490897, 0.665142, 0.799729

5.



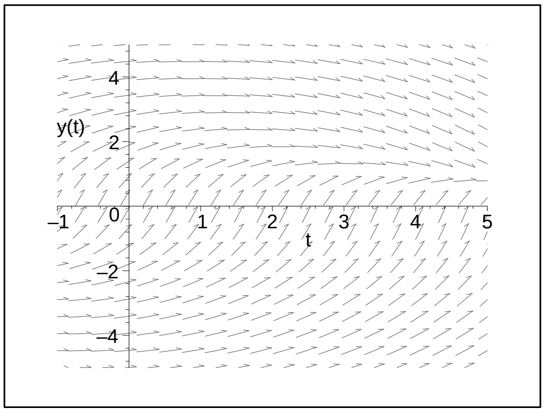
The solutions converge for $y \geq 0$. Solutions are undefined for $y < 0$.

6.



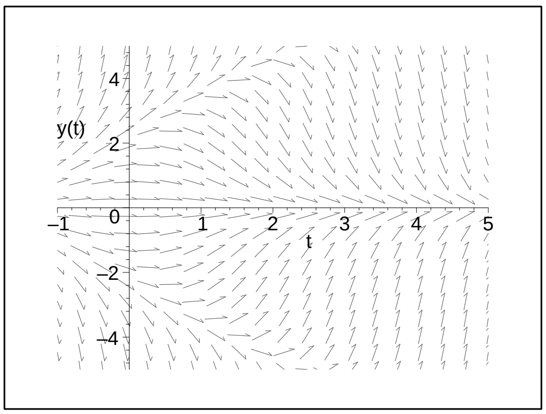
The solutions converge for $y \geq 0$. They diverge for $y < 0$.

7.



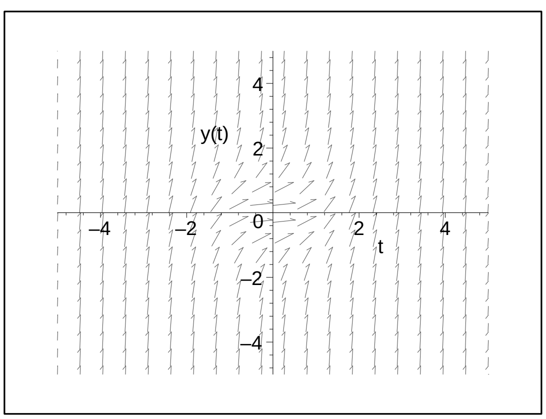
All solutions converge.

8.



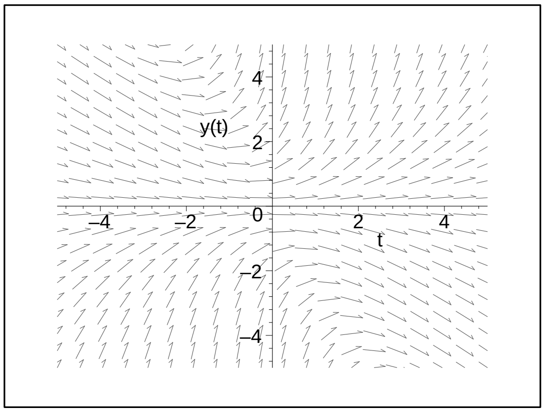
The solutions converge for $|y(0)| < 2.37$ (approximately). They diverge otherwise.

9.



All solutions diverge.

10.



All solutions diverge.

11. The Euler formula is $y_{n+1} = y_n - 3h\sqrt{y_n} + 5h$. The initial value is $y_0 = 2$.

(a) 2.30800, 2.49006, 2.60023, 2.66773, 2.70939, 2.73521

(b) 2.30167, 2.48263, 2.59352, 2.66227, 2.70519, 2.73209

(c) 2.29864, 2.47903, 2.59024, 2.65958, 2.70310, 2.73053

(d) 2.29686, 2.47691, 2.58830, 2.65798, 2.70185, 2.72959

12. The Euler formula is $y_{n+1} = (1 + 3h)y_n - ht_n y_n^2$. The initial value is $(t_0, y_0) = (0, 0.5)$.

(a) 1.70308, 3.06605, 2.44030, 1.77204, 1.37348, 1.11925

(b) 1.79548, 3.06051, 2.43292, 1.77807, 1.37795, 1.12191

(c) 1.84579, 3.05769, 2.42905, 1.78074, 1.38017, 1.12328

(d) 1.87734, 3.05607, 2.42672, 1.78224, 1.38150, 1.12411

13. The Euler formula is $y_{n+1} = y_n + h\frac{(4-t_n y_n)}{(1+y_n^2)}$ with $(t_0, y_0) = (0, -2)$.

(a) −1.48849, −0.412339, 1.04687, 1.43176, 1.54438, 1.51971

(b) −1.46909, −0.287883, 1.05351, 1.42003, 1.53000, 1.50549

(c) −1.45865, −0.217545, 1.05715, 1.41486, 1.52334, 1.49879

(d) −1.45212, −0.173376, 1.05941, 1.41197, 1.51949, 1.49490

14. The Euler formula is $y_{n+1} = (1 - ht_n)y_n + hy_n^3/10$, with $(t_0, y_0) = (0, 1)$.

(a) 0.950517, 0.687550, 0.369188, 0.145990, 0.0421429, 0.00872877

(b) 0.938298, 0.672145, 0.362640, 0.147659, 0.0454100, 0.0104931

(c) 0.932253, 0.664778, 0.359567, 0.148416, 0.0469514, 0.0113722

(d) 0.928649, 0.660463, 0.357783, 0.148848, 0.0478492, 0.0118978

15. The Euler formula is $y_{n+1} = y_n + h3t_n^2/(3y_n^2 - 4)$ with initial value $(t_0, y_0) = (1, 0)$.

(a) −0.166134, −0.410872, −0.804660, 4.15867

(b) −0.174652, −0.434238, −0.889140, −3.09810

(c) Since the line tangent to the solution is parallel to the $y-$axis when $y \cong \pm 1.155$, Euler's formula can be off by quite a bit. As the slope tends to ∞, using that slope as an approximation to the change in the function can cause a large error in the approximation.

16. The Euler formula is $y_{n+1} = y_n + h(t_n^2 + y_n^2)$ with $(t_0, y_0) = (0, 1)$. A reasonable estimate for y at $t = 0.8$ is between 5.5 and 6. No reliable estimate is possible at $t = 1$ from the specified data.

17. The Euler formula is $y_{n+1} = y_n + h(y_n^2 + 2t_n y_n)/(3 + t_n^2)$ with $(t_0, y_0) = (1, 2)$. A reasonable estimate for y at $t = 2.5$ is between 18 and 19. No reliable estimate is possible at $t = 3$ from the specified data.

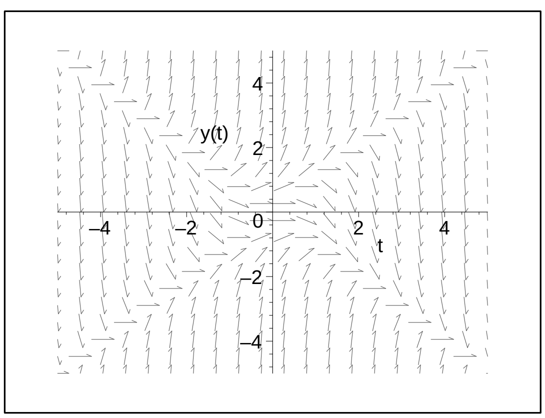
18.

(a) See the direction field in exercise 8 above.

(b) The Euler formula is $y_{n+1} = y_n + h(-t_n y_n + 0.1y_n^3)$. For $y_0 < 2.37$, the solutions seem to converge, while the solutions seem to diverge if $y_0 > 2.38$. We conclude that $2.37 < \alpha_0 < 2.38$

19.

(a)



(b) The Euler formula is $y_{n+1} = y_n + h(y_n^2 - t_n^2)$. For $y_0 < 0.67$, the solutions seem to converge, while the solutions seem to diverge if $y_0 > 0.68$. Therefore, we conclude that $0.67 < \alpha_0 < 0.68$

Section 1.4

1. The differential equation is second order, since the highest derivative in the equation is of order two. The equation is linear since the left hand side is a linear function of y and its derivatives and the right hand side is just a function of t.

2. The differential equation is second order since the highest derivative in the equation is of order two. The equation is nonlinear because of the term $y^2 d^2y/dt^2$.

3. The differential equation is fourth order since the highest derivative in the equation is of order four. The equation is linear since the left hand side is a linear function of y and its derivatives and the right hand side does not depend on y.

4. The differential equation is first order since the only derivative in the equation is of order one. The equation is nonlinear because of the y^2 term.

5. The differential equation is second order since the highest derivative in the equation is of order two. The equation is nonlinear because of the term $\sin(t+y)$ which is not a linear function of y.

6. The differential equation is third order since the highest derivative in the equation is of order three. The equation is linear because the left hand side is a linear function of y and its derivatives, and the right hand side is only a function of t.

7. $y_1(t) = e^t \implies y_1' = e^t \implies y_1'' = e^t$. Therefore, $y_1'' - y_1 = 0$. Also, $y_2 = \cosh t \implies y_2' = \sinh t \implies y_2' = \cosh t$. Therefore, $y_2'' - y_2 = 0$.

8. $y_1 = e^{-3t} \implies y_1' = -3e^{-3t} \implies y_1'' = 9e^{-3t}$. Therefore, $y_1''+2y_1'-3y_1 = (9-6-3)y_1 = 0$. Also, $y_2 = e^t \implies y_2' = y_2'' = e^t$. Therefore, $y_1'' + 2y_1' - 3y_1 = (1+2-3)e^t = 0$.

9. $y = 3t + t^2 \implies y' = 3 + 2t$. Therefore, $ty' - y = t(3+2t) - (3t+t^2) = t^2$.

10. $y_1 = t/3 \implies y_1' = 1/3 \implies y_1'' = y_1''' = y_1'''' = 0$. Therefore, $y_1'''' + 4y_1''' + 3y = t$. Also, $y_2 = e^{-t} + t/3 \implies y_2' = -e^{-t} + 1/3 \implies y_2'' = e^{-t} \implies y_2''' = -e^{-t} \implies y_2'''' = e^{-t}$. Therefore, $y_2'''' + 4y_2''' + 3y = e^{-t} - 4e^{-t} + 3(e^{-t} + t/3) = t$.

11. $y_1 = t^{1/2} \implies y_1' = t^{-1/2}/2 \implies y_1'' = -t^{-3/2}/4$. Therefore, $2t^2y_1'' + 3ty_1' - y_1 = 2t^2(-t^{-3/2}/4) + 3t(t^{-1/2}/2) - t^{1/2} = (-1/2 + 3/2 - 1)t^{1/2} = 0$. Also, $y_2 = t^{-1} \implies y_2' = -t^{-2} \implies y_2'' = 2t^{-3}$. Therefore, $2t^2y_2'' + 3ty_2' - y_2 = 2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} = (4-3-1)t^{-1} = 0$.

12. $y_1 = t^{-2} \implies y_1' = -2t^{-3} \implies y_1'' = 6t^{-4}$. Therefore, $t^2y_1'' + 5ty_1' + 4y_1 = t^2(6t^{-4}) + 5t(-2t^{-3}) + 4t^{-2} = (6-10+4)t^{-2} = 0$. Also, $y_2 = t^{-2}\ln t \implies y_2' = t^{-3} - 2t^{-3}\ln t \implies y_2'' = -5t^{-4} + 6t^{-4}\ln t$. Therefore, $t^2y_2'' + 5ty_2' + 4y_2 = t^2(-5t^{-4} + 6t^{-4}\ln t) + 5t(t^{-3} - 2t^{-3}\ln t) + 4(t^{-2}\ln t) = (-5+5)t^{-2} + (6-10+4)t^{-2}\ln t = 0$.

13. $y = (\cos t)\ln\cos t + t\sin t \implies y' = -(\sin t)\ln\cos t + t\cos t \implies y'' = -(\cos t)\ln\cos t - t\sin t + \sec t$. Therefore, $y'' + y = -(\cos t)\ln\cos t - t\sin t + \sec t + (\cos t)\ln\cos t + t\sin t = \sec t$.

14. $y = e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2} \implies y' = 2te^{t^2}\int_0^t e^{-s^2}\,ds + 1 + 2te^{t^2}$. Therefore, $y' - 2ty = 2te^{t^2}\int_0^t e^{-s^2}\,ds + 1 + 2te^{t^2} - 2t(e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}) = 1$.

15. Let $y = e^{rt}$. Then $y' = re^{rt}$. Substituting these terms into the differential equation, we have $y' + 2y = re^{rt} + 2e^{rt} = (r+2)e^{rt} = 0$. This equation implies $r = -2$.

16. Let $y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Substituting these terms into the differential equation, we have $y'' - y = (r^2 - 1)e^{rt} = 0$. This equation implies $r = \pm 1$.

17. Let $y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Substituting these terms into the differential equation, we have $y'' + y' - 6y = (r^2 + r - 6)e^{rt} = 0$. In order for r to satisfy this equation, we need $r^2 + r - 6 = 0$. That is, we need $r = 2, -3$.

18. Let $y = e^{rt}$. Then $y' = re^{rt}$, $y'' = r^2e^{rt}$ and $y''' = r^3e^{rt}$. Substituting these terms into the differential equation, we have $y''' - 3y'' + 2y' = (r^3 - 3r^2 + 2r)e^{rt} = 0$. In order for r to satisfy this equation, we need $r^3 - 3r^2 + 2r = 0$. That is, we need $r = 0, 1, 2$.

19. Let $y = t^r$. Then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these terms into the differential equation, we have $t^2y'' + 4ty' + 2y = t^2(r(r-1)t^{r-2}) + 4t(rt^{r-1}) + 2t^r = (r(r-1) + 4r + 2)t^r = 0$. In order for r to satisfy this equation, we need $r(r-1) + 4r + 2 = 0$. Simplifying this expression, we need $r^2 + 3r + 2 = 0$. The solutions of this equation are $r = -1, -2$.

20. Let $y = t^r$. Then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these terms into the differential equation, we have $t^2y'' - 4ty' + 4y = t^2(r(r-1)t^{r-2}) - 4t(rt^{r-1}) + 4t^r = (r(r-1) - 4r + 4)t^r = 0$. In order for r to satisfy this equation, we need $r(r-1) - 4r + 4 = 0$. Simplifying this expression, we need $r^2 - 5r + 4 = 0$. The solutions of this equation are $r = 1, 4$.

21.

(a) Consider Figure 1.4.1 in the text. There are two main forces acting on the mass: (1) the tension in the rod and (2) gravity. The tension, T, acts on the mass along the direction of the rod. By extending a line below and to the right of the mass at an angle θ with the vertical, we see that there is a force of magnitude $mg\cos\theta$ acting on the mass in that direction. Then extending a line below the mass and to the left, making an angle of $\pi - \theta$, we see the force acting on the mass in the tangential direction is $mg\sin\theta$.

(b) Newton's Second Law states that $\sum \mathbf{F} = m\mathbf{a}$. In the tangential direction, the equation of motion may be expressed as $\sum F_\theta = ma_\theta$. The tangential acceleration, a_θ is the linear acceleration along the path. That is, $a_\theta = Ld^2\theta/dt^2$. The only force acting in the tangential direction is the gravitational force in the tangential direction which is given by $-mg\sin\theta$. Therefore, $-mg\sin\theta = mLd^2\theta/dt^2$.

(c) Rearranging terms, we have

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta.$$

22.

(a) The kinetic energy of a particle of mass m is given by $T = \frac{1}{2}mv^2$ where v is its speed. A particle in motion on a circle of radius L has speed $L(d\theta/dt)$ where $d\theta/dt$ is its angular speed. Therefore,

$$T = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2.$$

(b) The potential energy of a particle is given by $V = mgh$ where h is the height above some point. Here we measure h as the height above the pendulum's lowest position. Since $\frac{L-h}{L} = \cos\theta$, we have $h = L(1-\cos\theta)$. Therefore,

$$V = mgL(1-\cos\theta).$$

(c) Since the total energy is conserved. We know that for $E = T + V$, $dE/dt = 0$. Here,

$$E = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1-\cos\theta)$$

implies

$$\frac{dE}{dt} = mL^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgL\sin\theta\frac{d\theta}{dt} = 0.$$

Simplifying this equation, we conclude that

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

23.

(a) Angular momentum is the moment about a certain point of linear momentum, which is given by

$$mv = mL\frac{d\theta}{dt}.$$

The moment about a pivot point is given by

$$M_p = mL^2\frac{d\theta}{dt}.$$

(b) The moment of the gravitational force is

$$M_g = -mg \cdot L\sin\theta.$$

Then $dM_p/dt = M_g$ implies

$$mL^2\frac{d^2\theta}{dt^2} = -mgL\sin\theta.$$

Rewriting this equation, we have

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$