CHAPTER

9

Nonlinear Differential Equations and Stability

9.1

For Problems 1 through 16, once the eigenvalues have been found, Table 9.1.1 will, for the most part, quickly yield the type of critical point and the stability. In all cases it can be easily verified that $\mathbf{A}$ is nonsingular.

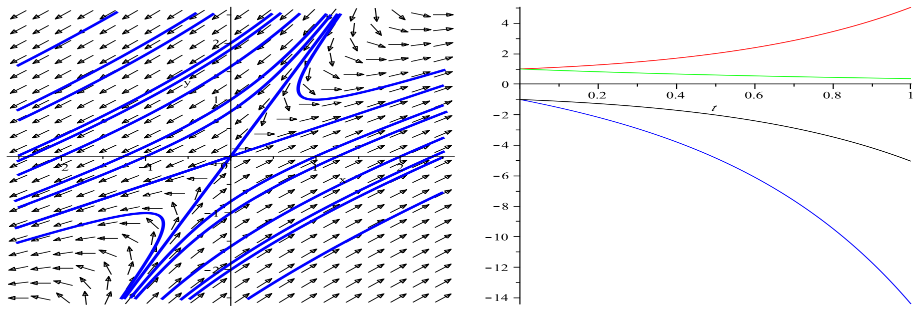
1.(a) Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $\det(\mathbf{A} - r\,\mathbf{I}) = r^2 - r - 2 = 0$. The roots of the characteristic equation are $r_1 = -1$ and $r_2 = 2$. For $r = -1$, the system of equations reduces to $4\xi_1 = 2\xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,2)^T$. Substitution of $r = 2$ results in the single equation $\xi_1 = 2\xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (2\,,1)^T$.

(b) The eigenvalues are real, with $r_1 r_2 < 0$. Hence the critical point is an unstable saddle point.

(c,d)



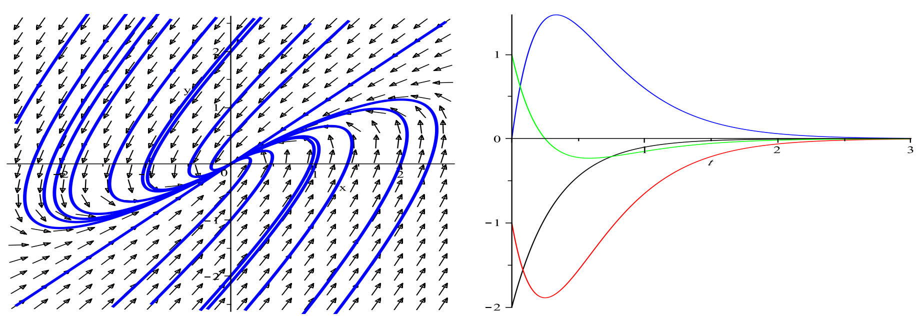
4.(a) Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 6r + 9 = 0$. The single root of the characteristic equation is $r = -3$. Setting $r = -3$, the components of the solution vector must satisfy $\xi_1 = \xi_2$. A corresponding eigenvector is $\boldsymbol{\xi} = (1\,,1)^T$.

(b) Since there is only one linearly independent eigenvector, the critical point is an asymptotically stable improper node. If we had found that there were two independent eigenvectors, then $(0,0)$ would have been a proper node, as indicated in Case 3a.

(c,d)



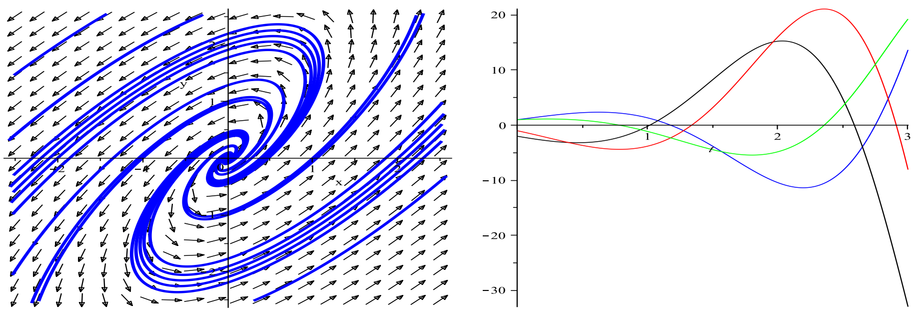
7.(a) Setting $\mathbf{x}= \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 3-r & -2 \\ 4 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 2r + 5 = 0$. The roots of the characteristic equation are $r = 1 \pm 2i$. Substituting $r = 1 - 2i$, the two equations reduce to $(1+i)\xi_1 - \xi_2 = 0$. The two eigenvectors are $\boldsymbol{\xi}^{(1)} = (1\,,1+i)^T$ and $\boldsymbol{\xi}^{(2)} = (1\,,1-i)^T$.

(b) The eigenvalues are complex conjugates, with positive real part. Hence the origin is an unstable spiral.

(c,d)



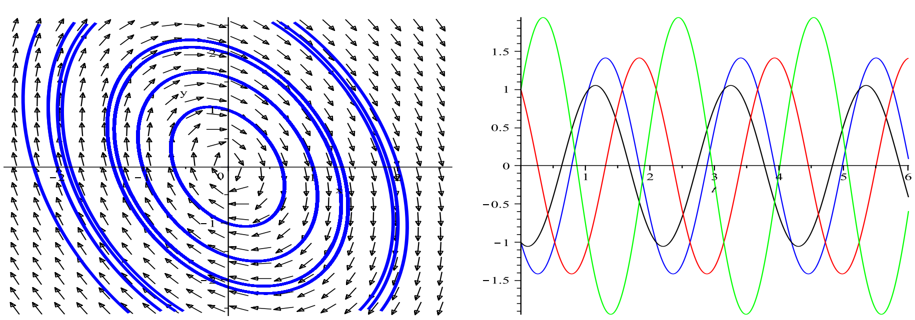
10.(a) The characteristic equation is given by

$$\begin{vmatrix} 1-r & 2 \\ -5 & -1-r \end{vmatrix} = r^2 + 9 = 0\,.$$

The equation has complex roots $r_{1,2} = \pm 3i$. For $r = -3i$, the components of the solution vector must satisfy $5\,\xi_1 + (1-3i)\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1-3i\,, -5)^T$. Substitution of $r = 3i$ results in $5\,\xi_1 + (1+3i)\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1+3i\,, -5)^T$. (These eigenvectors are complex constant multiples of the ones given in the text.)

(b) The eigenvalues are purely imaginary, hence the critical point is a center, which is stable.

(c,d)



13. If we let $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$, then $\mathbf{x}' = \mathbf{u}'$ and thus the system becomes

$$\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{x}^0 + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{u} - \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

which will be in the form of Eq.(2) if

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{x}^0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Using row operations, we find $\mathbf{x}^0 = (1,1)^T$. With the change of dependent variable, $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$, the differential equation can be written as

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{u}.$$

The critical point for the transformed equation is the origin. Setting $\mathbf{u} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 \\ 1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $\det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 2 = 0$. The roots of the characteristic equation are $r = \pm\sqrt{2}$. Hence the critical point is an unstable saddle point.

17.The equivalent system is $dx/dt = y$, $dy/dt = -(k/m)x - (c/m)y$, which is written in the form of Eq.(2) as

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -k/m & -c/m \end{pmatrix} \mathbf{x}.$$

The point $(0,0)$ is clearly a critical point, and since $\mathbf{A}$ is nonsingular, it is the only one. The characteristic equation is $r^2 + (c/m)r + k/m = 0$, with roots $r_{1,2} = (-c \pm \sqrt{c^2 - 4km})/2m$. In the underdamped case $c^2 - 4km < 0$, the characteristic roots are complex with negative real parts (since $c > 0$), and thus the critical point is an asymptotically stable spiral point. In the overdamped case $c^2 - 4km > 0$, the characteristic roots are real, unequal, and negative and hence the critical point is an asymptotically stable node. In the critically damped case $c^2 - 4km = 0$, the characteristic roots are equal and negative. As indicated in the solution of Problem 4, to determine whether this is an improper or proper node we must determine whether there are one or two linearly independent eigenvectors. We find only one eigenvector in this case, so the critical point $(0,0)$ is an asymptotically stable improper node.

18.(a) If $\mathbf{A}$ has one zero eigenvalue, then for $r = 0$ we get $\det(\mathbf{A} - r\mathbf{I}) = \det(\mathbf{A}) = 0$.

(b) Clearly, $\mathbf{x} = \mathbf{0}$ is a critical point. Also, part (a) shows $\mathbf{A}$ is singular which means $\mathbf{Ax} = \mathbf{0}$ has infinitely many solutions and consequently there are infinitely many critical points. Since $\mathbf{A}$ is a 2×2 matrix, the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ will yield the solution $x_2 = cx_1$, which indicates that the critical points lie on a straight line through the origin.

(c) From Chapter 7, the solution is $\mathbf{x}(t) = c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t}$. Now if $c_2 = 0$, then $\mathbf{x}(t) = c_1\boldsymbol{\xi}^{(1)}$ is a constant solution, so the critical points lie on the line with direction vector $\boldsymbol{\xi}^{(1)}$, just as the figure indicates. When $c_2 =\neq 0$, then $\mathbf{x}(t) = c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t}$, which gives the parametric equation of a half-line with direction vector $\boldsymbol{\xi}^{(2)}$, going through the critical point $c_1\boldsymbol{\xi}^{(1)}$. Thus we established that the trajectories follow the behavior indicated on the figure. When $r_2 < 0$, solutions converge toward the critical point as $t \to \infty$, when $r_2 > 0$, solutions diverge as $t \to \infty$.

19.(a) In this case, $\det(\mathbf{A} - r\mathbf{I}) = r^2 - (a_{11} + a_{22})r + a_{11}a_{22} - a_{21}a_{12} = 0$. Thus the roots are

$$r_{1,2} = \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{21}a_{12})}}{2}.$$

To obtain purely imaginary roots, we clearly need that $a_{11} + a_{22} = 0$, and then of course $r^2 = -(a_{11}a_{22} - a_{21}a_{12}) < 0$ if and only if $a_{11}a_{22} - a_{21}a_{12} > 0$.

(b) Eq.(i) can be written as $dx/dt = a_{11}x + a_{12}y$ and $dy/dt = a_{21}x + a_{22}y$, which gives Eq.(iii). Eq.(iii) can be rewritten as $(a_{21}x + a_{22}y)dx - (a_{11}x + a_{12}y)dy = 0$, which is exact since $a_{22} = -a_{11}$ from Eq.(ii).

(c) Integrating $\phi_x = a_{21}x + a_{22}y$, we get $\phi = a_{21}x^2/2 + a_{22}xy + g(y)$ and thus $\phi_y = a_{22}x + g' = -a_{11}x - a_{12}y$, so $g'(y) = -a_{12}y$ using Eq.(ii). Hence $\phi(x, y) = a_{21}x^2/2 + a_{22}xy - a_{12}y^2/2 = k/2$ is the solution to Eq.(iii). The quadratic equation $Ax^2 + Bxy + Cy^2 = D$ is an ellipse provided $B^2 - 4AC < 0$. Hence for our problem if $a_{22}^2 + a_{21}a_{12} < 0$ then Eq.(iv) is an ellipse. From $a_{11} + a_{22} = 0$ we have $a_{22}^2 = -a_{11}a_{22}$ and hence the condition becomes $-a_{11}a_{22} + a_{21}a_{12} < 0$ which is true by Eq.(ii). Thus Eq.(iv) is an ellipse under the conditions of Eq.(ii).

20. The system of ODEs can be written as

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathbf{x}.$$

The characteristic equation is $r^2 - p\,r + q = 0$. The roots are given by

$$r_{1,2} = \frac{p \pm \sqrt{p^2 - 4q}}{2} = \frac{p \pm \sqrt{\Delta}}{2}.$$

The results can be verified using Table 9.1.1.

22.(a) $\det(\mathbf{A} - r\,\mathbf{I}) = r^2 - r + 5/2 = 0$. The roots of the characteristic equation are $r = 1/2 \pm 3i/2$.

(b) Substituting $r = 1/2 + 3i/2$, the equations reduce to $(3 - 3i)\xi_1 - 5\,\xi_2 = 0$. Therefore a corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (5\,, 3 - 3i)^T$.

(c) We compute, using $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$:

$$\mathbf{y}' = \mathbf{P}^{-1}\mathbf{x}' = \mathbf{P}^{-1}\mathbf{A}\mathbf{x} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{y} = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})\mathbf{y}.$$

(d) We know that

$$\mathbf{P} = \begin{pmatrix} 5 & 0 \\ 3 & -3 \end{pmatrix}, \text{ so } \mathbf{P}^{-1} = \frac{1}{15}\begin{pmatrix} 3 & 0 \\ 3 & -5 \end{pmatrix}.$$

Thus

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \frac{1}{15}\begin{pmatrix} 3 & 0 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} 2 & -5/2 \\ 9/5 & -1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 3 & -3 \end{pmatrix} =$$

$$= \frac{1}{15}\begin{pmatrix} 6 & -15/2 \\ -3 & -5/2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 3 & -3 \end{pmatrix} = \frac{1}{15}\begin{pmatrix} 15/2 & 45/2 \\ -45/2 & 15/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 3/2 \\ -3/2 & 1/2 \end{pmatrix},$$

as claimed.

9.2

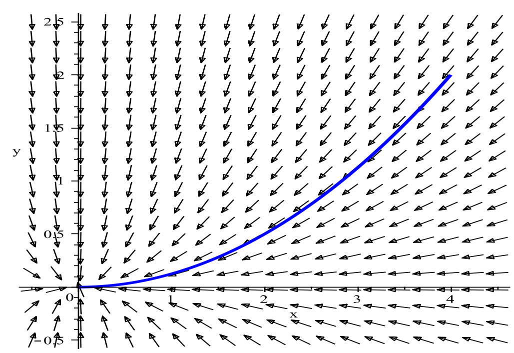
1. The differential equations can be combined to obtain a related ODE

$$\frac{dy}{dx} = \frac{2y}{x}.$$

The equation is separable, with

$$\frac{dy}{y} = \frac{2\,dx}{x}.$$

The solution is given by $y = C\,x^2$. Note that the system is uncoupled, and hence we also have $x = x_0 e^{-t}$ and $y = y_0 e^{-2t}$. Matching the initial conditions, we obtain $x(t) = 4e^{-t}$ and $y(t) = 2e^{-2t}$.



In order to determine the direction of motion along the trajectory, observe that for positive initial conditions, both x and y will decrease.

3. The trajectories of the system satisfy the ODE

$$\frac{dy}{dx} = -\frac{x}{y}.$$

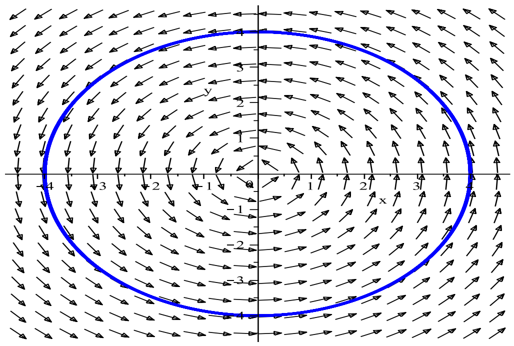
The equation is separable, with $y\,dy = -x\,dx$. Hence the trajectories are given by $x^2 + y^2 = C^2$, in which C is arbitrary. Evidently, the trajectories are circles. Invoking the initial conditions, we find that $C^2 = 16$ for both pairs. The system of ODEs can also be written as

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x}.$$

Using the methods in Chapter 7, it is easy to show that the eigenvalues are $\pm i$, and we obtain the two real solutions

$$\mathbf{u}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad \text{and} \quad \mathbf{v}(t) = \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}.$$

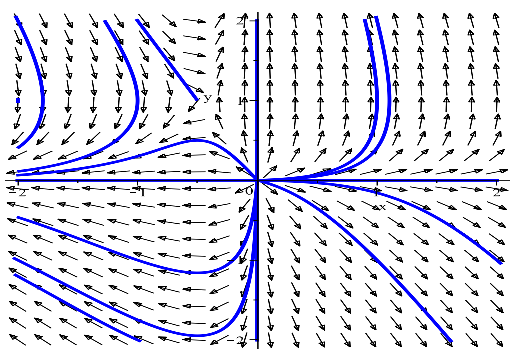
The first set of initial conditions then yields $x = 4\cos t$, $y = 4\sin t$, while the second set gives $x = -4\sin t$, $y = 4\cos t$.



The direction of motion is counterclockwise for both trajectories.

5.(a) The critical points are given by the solution set of the equations $x(1-y)=0$ and $y(1+2x)=0$. Clearly, $(0\,,0)$ is a solution. If $x \neq 0$, then $y=1$ and $x=-1/2$. Hence the critical points are $(0\,,0)$ and $(-1/2\,,1)$.

(b)

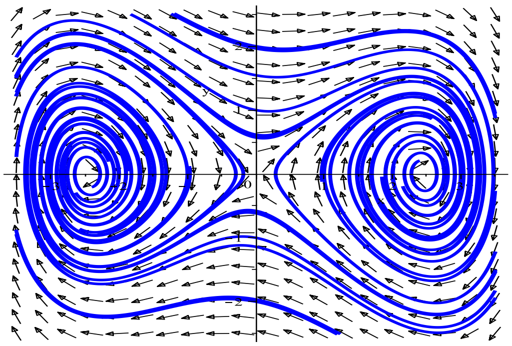


(c) Based on the phase portrait, all trajectories starting near the origin diverge. Hence the critical point $(0\,,0)$ is unstable. Examining the phase curves near the critical point $(-1/2\,,1)$, the equilibrium point has the properties of a saddle, and hence it is unstable.

(d) There is no basin of attraction since all critical points are unstable.

12.(a) The critical points are given by the solution set of the equations $y=0$ and $x - x^3/6 - y/5 = 0$. The first equation gives $y=0$, and then $x(1-x^2/6)=0$ gives the critical points $(0\,,0)$, $(\sqrt{6}\,,0)$, and $(-\sqrt{6}\,,0)$.

(b)



(c) We can see that $(\sqrt{6}\,, 0)$ and $(-\sqrt{6}\,, 0)$ are spiral points which are asymptotically stable. $(0, 0)$ is a saddle point, hence unstable.

(d) The basin of attraction is defined by the trajectory starting at $(0, 0)$, encircling each of the critical points $(\sqrt{6}\,, 0)$ and $(-\sqrt{6}\,, 0)$, and ending up at $(0, 0)$ again.

17. (a) The trajectories are solutions of the differential equation

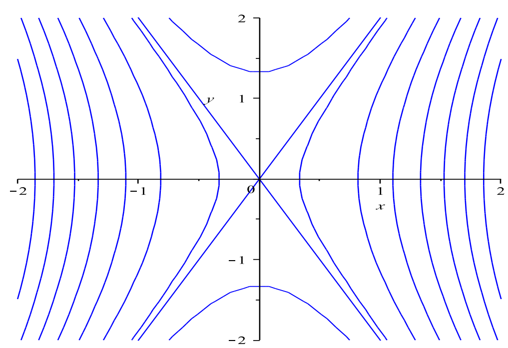
$$\frac{dy}{dx} = \frac{4x}{y},$$

which can also be written as $4x\,dx - y\,dy = 0$. Integrating, we obtain

$$4x^2 - y^2 = C.$$

Hence the trajectories are hyperbolas (for $C \neq 0$) and the straight lines $y = \pm 2x$ (for $C = 0$).

(b)



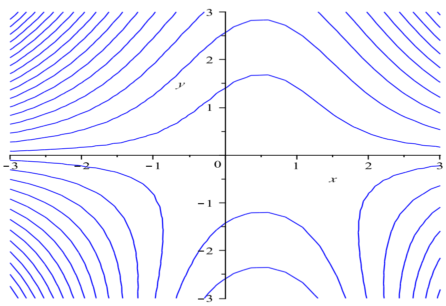
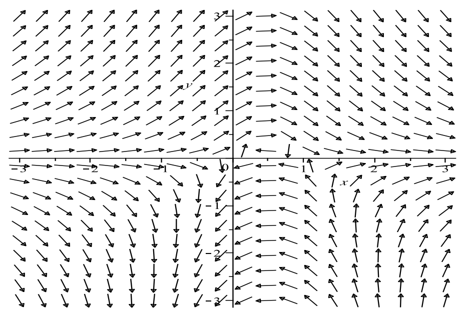
Based on the differential equations, in the first quadrant both x and y are increasing, in the second quadrant x is increasing and y is decreasing, in the third quadrant both x and y are decreasing, and in the fourth quadrant x is decreasing and y is increasing.

21.(a) The trajectories are solutions of the differential equation

$$\frac{dy}{dx} = \frac{y - 2xy}{-x + y + x^2},$$

which can also be written as $(y - 2xy)dx + (x - y - x^2)dy = 0$. The resulting ODE is exact, with $H_x = y - 2xy$ and $H_y = x - y - x^2$. Integrating the first equation, we find that $H(x\,,y) = xy - x^2y + f(y)$. It follows that $H_y = x - x^2 + f\,'(y)$. Comparing the two partial derivatives, we obtain $f(y) = -y^2/2 + c$. Hence $H(x\,,y) = xy - x^2y - y^2/2$.

(b) The associated direction field shows the direction of motion along the trajectories.

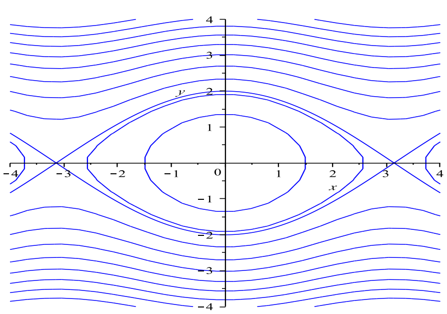
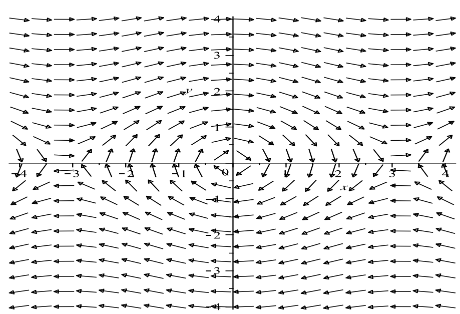



23.(a) The trajectories are solutions of the differential equation

$$\frac{dy}{dx} = \frac{-\sin x}{y},$$

so $y\,dy + \sin x\,dx = 0$ and thus $H(x, y) = y^2/2 - \cos x$.

(b) The associated direction field shows the direction of motion along the trajectories.

25.

$$\frac{d\Phi}{dt}(t) = \frac{d\phi}{dt}(t - s) = F(\phi(t - s), \psi(t - s)) = F(\Phi(t), \Psi(t))$$

and

$$\frac{d\Psi}{dt}(t) = \frac{d\psi}{dt}(t - s) = G(\phi(t - s), \psi(t - s)) = G(\Phi(t), \Psi(t)).$$

Therefore, $\Phi(t), \Psi(t)$ is a solution for $\alpha + s < t < \beta + s$.

26. Let C_0 be the trajectory generated by the solution $x = \phi_0(t)$, $y = \psi_0(t)$ with $\phi_0(t_0) = x_0$, $\psi_0(t_0) = y_0$ and let C_1 be the trajectory generated by the solution $x = \phi_1(t)$, $y = \psi_1(t)$ with $\phi_1(t_1) = x_0$, $\psi_1(t_1) = y_0$. From Problem 25, we know that $\Phi_1(t) = \phi_1(t - (t_0 - t_1))$, $\Psi_1(t) = \psi_1(t - (t_0 - t_1))$ is a solution. Further, $\Phi_1(t_0) = \phi_1(t_1) = x_0$ and $\Psi_1(t_0) = y_0$. Then, by uniqueness, $\phi_0(t) = \Phi_1(t)$ and $\psi_0(t) = \Phi_1(t)$. Therefore, the trajectories are the same.

27. From the existence and uniqueness theorem we know that if the two solutions $x = \phi(t)$, $y = \psi(t)$ and $x = x_0$, $y = y_0$ satisfy $\phi(a) = x_0$, $\psi(a) = y_0$ and $x = x_0$, $y = y_0$ at $t = a$, then these solutions are identical. Hence $\phi(t) = x_0$ and $\psi(t) = y_0$ for all t contradicting the fact that the trajectory generated by $\phi(t)$ and $\psi(t)$ started at a noncritical point.

28. Since the trajectory is closed, there is at least one point (x_0, y_0) such that $\phi(t_0) = x_0$, $\psi(t_0) = y_0$ and a number $T > 0$ such that $\phi(t_0 + T) = x_0$, $\psi(t_0 + T) = y_0$. From Problem 25, we know that $\Phi(t) = \psi(t + T)$, $\Psi(t) = \psi(t + T)$ will also be a solution. But, then by uniqueness $\Phi(t) = \phi(t)$ and $\Psi(t) = \psi(t)$ for all t. Therefore, $\phi(t + T) = \phi(t)$ and $\psi(t + T) = \psi(t)$ for all t. Therefore, the solution is periodic with period T.

9.3

In Problems 1 through 4, write the system in the form of Eq.(4). Then if $\mathbf{g}(0) = 0$ we may conclude that $(0, 0)$ is a critical point. In addition, if $\mathbf{g}$ satisfies Eq.(5) or Eq.(6), then the system is locally linear. In this case the linear system, Eq.(1), will determine, in most cases, the type and stability of the critical point $(0, 0)$ of the locally linear system. These results are summarized in Table 9.3.1.

3. In this case the system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (1+x)\sin y \\ 1 - \cos y \end{pmatrix}.$$

However, the coefficient matrix is singular and $g_1(x, y) = (1 + x)\sin y$ does not satisfy Eq.(6). However, we can see that $(0, 0)$ is a critical point. Consider now the Taylor series $\sin y = y - y^3/3! + \ldots$ and $\cos y = 1 - y^2/2! + \ldots$. The system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} xy - y^3/3 + \ldots \\ y^2/2! + \ldots \end{pmatrix}.$$

In this form, $\mathbf{g}$ satisfies Eq.(6). This means that considering the original system in the form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (1+x)\sin y - y \\ 1 - \cos y \end{pmatrix}$$

the linear part is nonsingular and $\mathbf{g}$ satisfies Eq.(6). The linear system has eigenvalues $\pm i$ and thus the origin is a center which is stable and the nonlinear system has either center or a spiral point at the origin and the stability is indeterminate, from Table 9.3.1.

4. The system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2 \\ 0 \end{pmatrix},$$

so

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{g} = \begin{pmatrix} y^2 \\ 0 \end{pmatrix}.$$

Since $\mathbf{g}(\mathbf{0}) = (0,0)^T$, we conclude that $(0,0)$ is a critical point. Following the procedure of Example 1, we let $x = r\cos\theta$ and $y = r\sin\theta$ and thus $g_1(x,y)/r = r^2\sin^2\theta/r \to 0$ as $r \to 0$ and thus the system is locally linear. The characteristic equation of the associated linear system is $(r-1)^2 = 0$, with equal roots $r_1 = r_2 = 1$. Since the roots are equal, we determine that there is only one corresponding eigenvector and thus the critical point for the linear system is an unstable improper node. From Table 9.3.1 we then conclude that the given system, which is locally linear, has a critical point near $(0,0)$ which is either a node or a spiral point (depending on how the roots bifurcate) which is unstable.

6.(a) The critical points consist of the solution set of the equations $x(1-x-y) = 0$, $y(3-x-2y) = 0$. Solutions are $x = 0$, $y = 0$; $x = 0$, $3-2y = 0$ i.e. $y = 3/2$; $y = 0$, $1-x = 0$ i.e. $x = 1$; and $1-x-y = 0$, $3-x-2y = 0$ which give $x = -1$, $y = 2$. Thus we found the four critical points $(0,0)$, $(0,3/2)$, $(1,0)$, $(-1,2)$.

(b) Here, we have $F(x,y) = x - x^2 - xy$ and $G(x,y) = 3y - xy - 2y^2$. Therefore, the Jacobian matrix for this system is

$$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1-2x-y & -x \\ -y & 3-x-4y \end{pmatrix}.$$

Therefore, near the critical point $(0,0)$, the Jacobian matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

and the corresponding linear system near $(0,0)$ is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Near the critical point $(0,3/2)$, the Jacobian matrix is

$$\begin{pmatrix} -1/2 & 0 \\ -3/2 & -3 \end{pmatrix}$$

and the corresponding linear system near $(0,3/2)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ -3/2 & -3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x$ and $v = y - 3/2$. Near the critical point $(1, 0)$, the Jacobian matrix is

$$\begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}$$

and the corresponding linear system near $(1, 0)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - 1$ and $v = y$. Near the critical point $(-1, 2)$, the Jacobian matrix is

$$\begin{pmatrix} 1 & 1 \\ -2 & -4 \end{pmatrix}$$

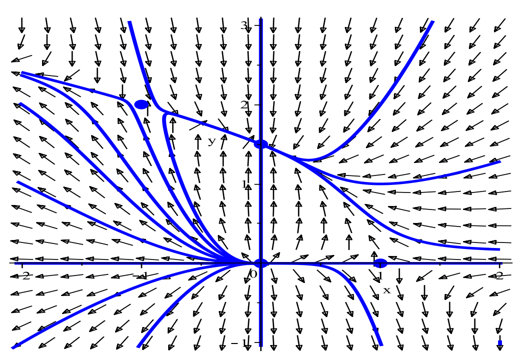
and the corresponding linear system near $(-1, 2)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x + 1$ and $v = y - 2$.

(c) The eigenvalues of the linear system near $(0, 0)$ are $\lambda = 1, 3$. From this, we can conclude that $(0, 0)$ is an unstable node for the nonlinear system. The eigenvalues of the linear system near $(0, 3/2)$ are $\lambda = -1/2, -3$. From this, we can conclude that $(0, 3/2)$ is an asymptotically stable node for the nonlinear system. The eigenvalues of the linear system near $(1, 0)$ are $\lambda = -1, 2$. From this, we can conclude that $(1, 0)$ is a saddle point for the nonlinear system. The eigenvalues of the linear system near $(-1, 2)$ are $\lambda = (-3 \pm \sqrt{17})/2$. From this, we can conclude that $(-1, 2)$ is an unstable saddle point for the nonlinear system.

(d)



10.(a) To find the critical points, we need to solve the equations $x + x^2 + y^2 = 0$ and $y - xy = y(1 - x) = 0$. Solving these equations, we find that the critical points are $(0, 0)$ and $(-1, 0)$.

(b) Here, we have $F(x,y) = x + x^2 + y^2$ and $G(x,y) = y - xy$. Therefore, the Jacobian matrix for this system is

$$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1+2x & 2y \\ -y & 1-x \end{pmatrix}.$$

Therefore, near the critical point $(0,0)$, the Jacobian matrix is

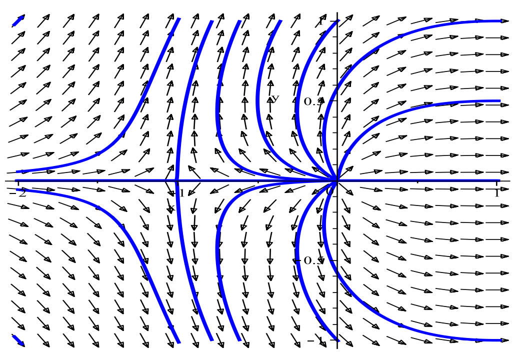
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Near the critical point $(-1,0)$, the Jacobian matrix is

$$\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

(c) The eigenvalues of the linear system near $(0,0)$ are $\lambda = 1$. From this, we can conclude that $(0,0)$ is an unstable node or spiral point for the nonlinear system, depending on how the roots bifurcate. The eigenvalues of the linear system near $(-1,0)$ are $\lambda = -1, 2$. From this, we can conclude that $(-1,0)$ is an unstable saddle point for the nonlinear system.

(d)



18.(a) The critical points occur when either $y = 1$ or $y = 2x$ and either $x = -2$ or $x = 2y$. Therefore, we see that the critical points are $(0,0)$, $(2,1)$, $(-2,1)$ and $(-2,-4)$.

(b) Here, we have $F(x,y) = (1-y)(2x-y)$ and $G(x,y) = (2+x)(x-2y)$. Therefore, the Jacobian matrix for this system is

$$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 2-2y & -2x-1+2y \\ 2+2x-2y & -4-2x \end{pmatrix}.$$

Therefore, near the critical point $(0,0)$, the Jacobian matrix is

$$\begin{pmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & -4 \end{pmatrix}$$

and the corresponding linear system near $(0,0)$ is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Near the critical point $(2,1)$, the Jacobian matrix is

$$\begin{pmatrix} F_x(2,1) & F_y(2,1) \\ G_x(2,1) & G_y(2,1) \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 4 & -8 \end{pmatrix}$$

and the corresponding linear system near $(2,1)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - 2$ and $v = y - 1$. Near the critical point $(-2,1)$, the Jacobian matrix is

$$\begin{pmatrix} F_x(-2,1) & F_y(-2,1) \\ G_x(-2,1) & G_y(-2,1) \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ -4 & 0 \end{pmatrix}$$

and the corresponding linear system near $(-2,1)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x + 2$ and $v = y - 1$. Near the critical point $(-2,-4)$, the Jacobian matrix is

$$\begin{pmatrix} F_x(-2,-4) & F_y(-2,-4) \\ G_x(-2,-4) & G_y(-2,-4) \end{pmatrix} = \begin{pmatrix} 10 & -5 \\ 6 & 0 \end{pmatrix}$$

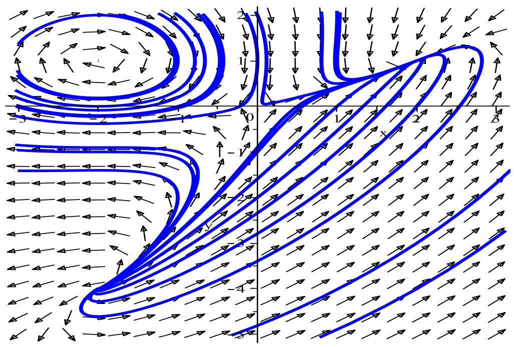
and the corresponding linear system near $(-2,-4)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 10 & -5 \\ 6 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x + 2$ and $v = y + 4$.

(c) The eigenvalues of the linear system near $(0,0)$ are $\lambda = -1 \pm \sqrt{7}$. From this, we can conclude that $(0,0)$ is an unstable saddle point for the nonlinear system. The eigenvalues of the linear system near $(2,1)$ are $\lambda = -2, -6$. From this, we can conclude that $(2,1)$ is an asymptotically stable node for the nonlinear system. The eigenvalues of the linear system near $(-2,1)$ are $\lambda = \pm 2\sqrt{5}i$. From this, we can only conclude that $(-2,1)$ is a either a center or a spiral point, and we cannot determine its stability. The eigenvalues of the linear system near $(-2,-4)$ are $\lambda = 5 \pm \sqrt{5}i$. From this, we can conclude that $(-2,-4)$ is an unstable spiral point.

(d)



20.(a) The critical points occur when $x = 0$ and $-2y + x^3 = 0$. Plugging the first equation into the second, we see that the only critical point is $(0, 0)$. The Jacobian matrix is given by

$$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3x^2 & -2 \end{pmatrix}.$$

Therefore, the coefficient matrix of the linearized system near $(0, 0)$ is

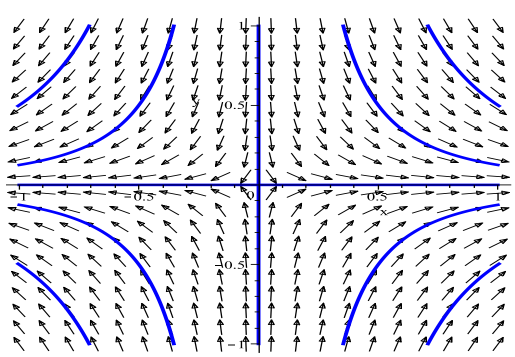
$$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}.$$

The eigenvalues are 1 and -2 and thus the origin is an unstable saddle point.

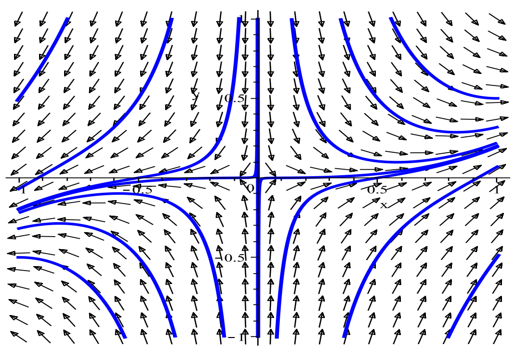
(b) The linear system is

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = -2y.$$

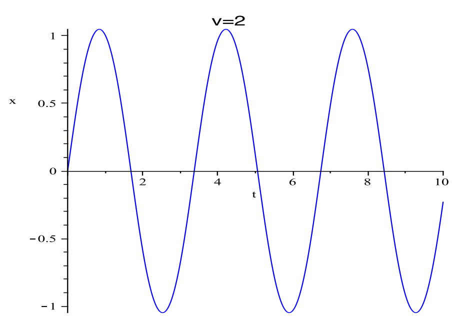
Thus, $x(t) = c_1 e^t$ and $y(t) = c_2 e^{-2t}$. To sketch these, solve the first equation for e^t and substitute into the second to obtain $y = c_1^2 c_2 / x^2$, $c_1 \neq 0$. Several trajectories are shown in the figure. Since $x(t) = c_1 e^t$, we must pick $c_1 = 0$ for $x \to 0$ as $t \to \infty$. Thus $x = 0$, $y = c_2 e^{-2t}$ (the vertical axis) is the only trajectory for which $x \to 0$, $y \to 0$ as $t \to \infty$.

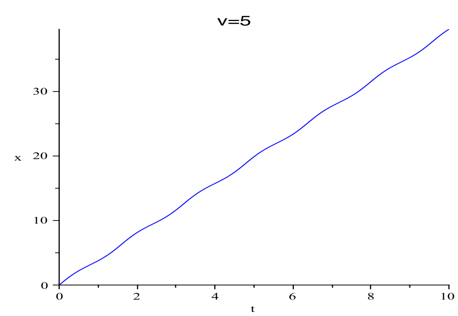


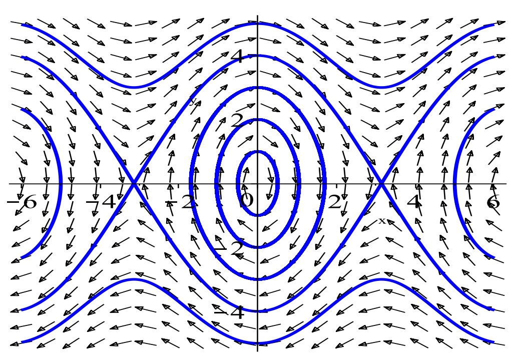
(c) For the nonlinear system, dy/dx is given by $dy/dx = (-2y + x^3)/x$. This equation can be rewritten as $(2y - x^3)dx + xdy = 0$. Multiplying this equation by x, we can rewrite the equation as $(2xy - x^4)dx + x^2dy = 0$. We notice that this equation is exact. Integrating $2xy - x^4$ with respect to x, we have $H(x, y) = x^2y - x^5/5 + h(y)$. Then differentiating with respect to y, we have $H_y = x^2 + h'(y) = x^2$. Therefore, the trajectories are level curves of $H(x, y) = x^2y - x^5/5$. We can see that the level curve $H = 0$ gives us $x = 0$ and $y = x^3/5$, which verifies the problem's statement. The trajectories are shown below:



24.(a)



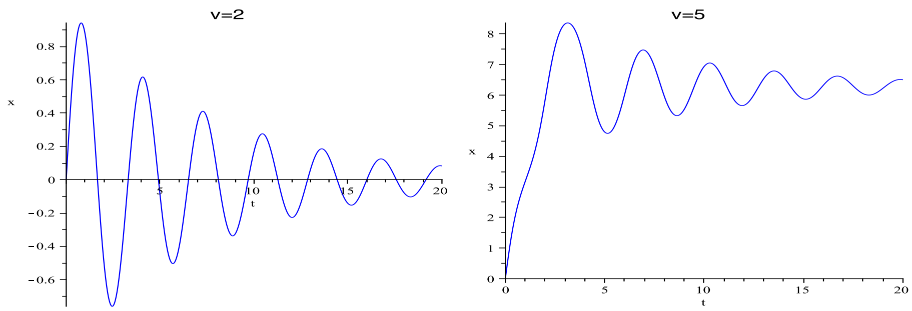


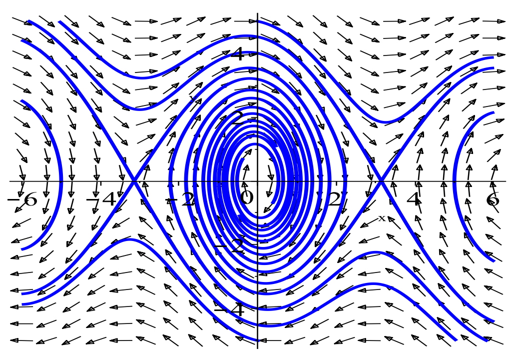


The graph for $v = 5$ indicates the initial velocity causes the pendulum to rotate beyond the upper critical point. Since there is no damping, the x value continues to increase indefinitely.

(b) From the graphs in part (a), we see that v_c is between $v = 2$ and $v = 5$. Using several values of v, we estimate $v_c \approx 4.00$. (See the phase plane figure above.)

25.(a)





For $v = 2$, the motion is damped oscillatory about $x = 0$. For $v = 5$ the pendulum swings all the way around once and then is a damped oscillation about $x = 2\pi$ (after one full rotation).

(b) From the graphs in part (a), we see that v_c is between $v = 2$ and $v = 5$. Using several values of v, we estimate $v_c \approx 4.52$. (See the phase plane figure above.)

29.(a) Setting $c = 0$ in Eq.(10) of Section 9.2 we obtain

$$mL^2 \frac{d^2\theta}{dt^2} + mgL \sin\theta = 0.$$

Considering $d\theta/dt$ as a function of θ and using the chain rule we have

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d}{d\theta}\left(\frac{d\theta}{dt}\right)\frac{d\theta}{dt} = \frac{1}{2}\frac{d}{d\theta}\left(\frac{d\theta}{dt}\right)^2.$$

Thus

$$\frac{1}{2}mL^2 \frac{d}{d\theta}\left(\frac{d\theta}{dt}\right)^2 = -mgL\sin\theta.$$

Now integrate both sides from α to θ where $d\theta/dt = 0$ at $\theta = \alpha$:

$$\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 = mgL(\cos\theta - \cos\alpha).$$

Thus $(d\theta/dt)^2 = (2g/L)(\cos\theta - \cos\alpha)$. Since we are releasing the pendulum with zero velocity from a positive angle α, the angle θ will initially be decreasing so $d\theta/dt < 0$. If we restrict our attention to the range of θ from $\theta = \alpha$ to $\theta = 0$, we can assert $d\theta/dt = -\sqrt{2g/L}\sqrt{\cos\theta - \cos\alpha}$. Solving for dt gives

$$dt = -\sqrt{\frac{L}{2g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

(b) Since there is no damping, the pendulum will swing from its initial angle α to 0 to $-\alpha$, then back through 0 again to the angle α in one period. It follows that $\theta(T/4) = 0$. Integrating the last equation and noting that as t goes from 0 to $T/4$, θ goes from α to 0 yields

$$\frac{T}{4} = -\sqrt{\frac{L}{2g}}\int_{\alpha}^{0}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

(c) Using the suggested substitutions and $(1/2)\cos(\theta/2)d\theta = (1/2)\sqrt{1 - k^2\sin^2\phi}\,d\theta = \sin(\alpha/2)\cos\phi\,d\phi$, we obtain

$$\frac{T}{4} = -\sqrt{\frac{L}{2g}}\int_{\alpha}^{0}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = -\sqrt{\frac{L}{2g}}\int_{\alpha}^{0}\frac{d\theta}{\sqrt{2\sin^2(\alpha/2) - 2\sin^2(\theta/2)}} =$$

$$= -\sqrt{\frac{L}{2g}}\int_{\pi/2}^{0}\frac{2\sin(\alpha/2)\cos\phi\,d\phi}{\sqrt{1 - k^2\sin^2\phi}\sqrt{2\sin^2(\alpha/2) - 2\sin^2(\alpha/2)\sin^2\phi}},$$

which gives

$$T = 4\sqrt{\frac{L}{g}}\int_{0}^{\pi/2}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}.$$

(d) Set $k = \sin(\alpha/2) = \sin(A/2)$ and $g/L = 4$.

30.(a) If $dx/dt = y$, then $d^2x/dt^2 = dy/dt = -g(x) - c(x)y$.

(b) Under the given assumptions we have $g(x) = g(0) + g'(0)x + g''(\xi_1)x^2/2$ and $c(x) = c(0) + c'(\xi_2)x$, where $0 < \xi_1, \xi_2 < x$ and $g(0) = 0$. Hence $dy/dt = -g'(0)x - c(0)y - (g''(\xi_1)x^2/2 - c'(\xi_2)xy)$ and thus the system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g'(0) & -c(0) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ -g''(\xi_1)x^2/2 - c'(\xi_2)xy \end{pmatrix},$$

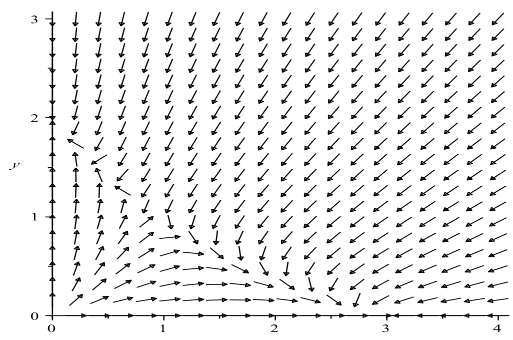
from which the results follow.

(c) In the first case, the trace is negative and the determinant is positive, so we have an asymptotically stable equilibrium at the origin. In the second case, the determinant is negative, so we have a saddle point at the origin, which is unstable.

9.4

For Problems 1 through 6, when the Jacobian is evaluated at a critical point, the corresponding linear systems are of the form $\mathbf{u}' = \mathbf{J}\mathbf{u}$. Their character is analyzed similarly to Example 1.

3. (a)



(b) The critical points are solutions of the system

$$x(1.5 - 0.5x - y) = 0$$
$$y(2 - y - 1.125x) = 0.$$

The four critical points are $(0,0)$, $(0,2)$, $(3,0)$ and $(4/5, 11/10)$.

(c) The Jacobian matrix is

$$\mathbf{J} = \begin{pmatrix} 3/2 - x - y & -x \\ -1.125y & 2 - 2y - 1.125x \end{pmatrix}.$$

At $(0,0)$,

$$\mathbf{J}(0,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The associated eigenvalues and eigenvectors are $r_1 = 3/2$, $\boldsymbol{\xi}_1 = (1,0)^T$ and $r_2 = 2$, $\boldsymbol{\xi}_2 = (0,1)^T$. The eigenvalues are positive. Therefore, the origin is an unstable node.
At $(0,2)$,

$$\mathbf{J}(0,2) = \begin{pmatrix} -1/2 & 0 \\ -9/4 & -2 \end{pmatrix}.$$

The associated eigenvalues and eigenvectors are $r_1 = -1/2$, $\boldsymbol{\xi}_1 = (1, -3/2)^T$ and $r_2 = -2$, $\boldsymbol{\xi}_2 = (0, 1)^T$. The eigenvalues are both negative. Therefore, $(0, 2)$ is a a stable node, which is asymptotically stable.
At $(3, 0)$,

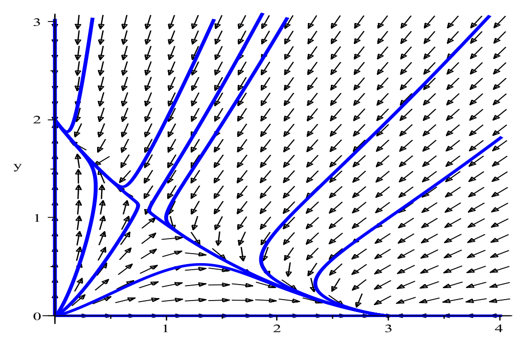
$$\mathbf{J}(3,0) = \begin{pmatrix} -3/2 & -3 \\ 0 & -11/8 \end{pmatrix}.$$

The associated eigenvalues and eigenvectors are $r_1 = -3/2$, $\boldsymbol{\xi}_1 = (1, 0)^T$ and $r_2 = -11/8$, $\boldsymbol{\xi}_2 = (-24, 1)^T$. The eigenvalues are both negative. Therefore, this critical point is a stable node, which is asymptotically stable.
At $(4/5, 11/10)$,

$$\mathbf{J}(4/5, 11/10) = \begin{pmatrix} -2/5 & -4/5 \\ -99/80 & -11/10 \end{pmatrix}.$$

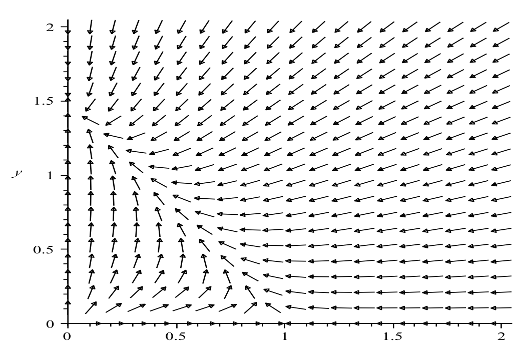
The associated eigenvalues and eigenvectors are $r_1 = -3/4 + \sqrt{445}/20$, $\boldsymbol{\xi}_1 = (1, (7 - \sqrt{445})/16)^T$ and $r_2 = -3/4 - \sqrt{445}/20$, $\boldsymbol{\xi}_2 = (0, (7 + \sqrt{445})/16)^T$. The eigenvalues are of opposite sign. Therefore, $(4/5, 11/10)$ is a saddle, which is unstable.

(d,e)



(f) As in Example 2, one species will die out, depending on the initial conditions. For an initial condition lying below the separatrix, the species denoted by x will survive, while if the initial condition is above the separatrix the species denoted by y will survive.

5.(a)



(b) The critical points are solutions of the system

$$x(1-x-y)=0$$
$$y(1.5-y-x)=0.$$

The three critical points are $(0,0)$, $(0,3/2)$ and $(1,0)$.

(c) The Jacobian matrix is

$$\mathbf{J}=\begin{pmatrix}1-2x-y & -x\\ -y & 1.5-2y-x\end{pmatrix}.$$

At $(0,0)$,

$$\mathbf{J}(0,0)=\begin{pmatrix}1 & 0\\ 0 & 1.5\end{pmatrix}.$$

The associated eigenvalues and eigenvectors are $r_1=1$, $\boldsymbol{\xi}_1=(1,0)^T$ and $r_2=1.5$, $\boldsymbol{\xi}_2=(0,1)^T$. The eigenvalues are positive. Therefore, the origin is an unstable node.
At $(0,3/2)$,

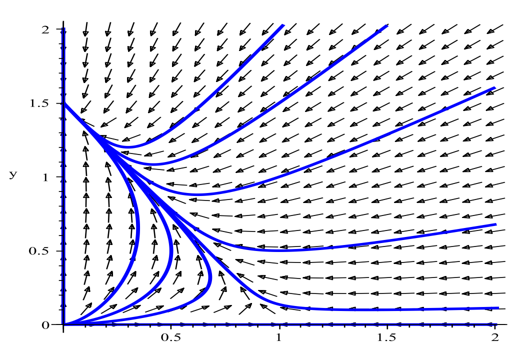
$$\mathbf{J}(0,3/2)=\begin{pmatrix}-1/2 & 0\\ -3/2 & -3/2\end{pmatrix}.$$

The associated eigenvalues and eigenvectors are $r_1=-1/2$, $\boldsymbol{\xi}_1=(1,-3/2)^T$ and $r_2=-3/2$, $\boldsymbol{\xi}_2=(0,1)^T$. The eigenvalues are both negative. Therefore, $(0,3/2)$ is a stable node, which is asymptotically stable.
At $(1,0)$,

$$\mathbf{J}(1,0)=\begin{pmatrix}-1 & -1\\ 0 & 1/2\end{pmatrix}.$$

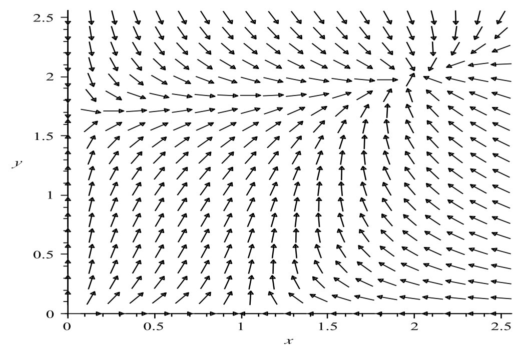
The associated eigenvalues and eigenvectors are $r_1=-1$, $\boldsymbol{\xi}_1=(1,0)^T$ and $r_2=1/2$, $\boldsymbol{\xi}_2=(1,-3/2)^T$. The eigenvalues are of opposite sign. Therefore, this critical point is a saddle, which is unstable.

(d,e)



(f) All trajectories converge to the stable node $(0,1.5)$, thus only one species survives.

6.(a)



(b) The critical points are solutions of the system

$$x(1 - x + 0.5y) = 0$$
$$y(2.5 - 1.5y + 0.25x) = 0.$$

The four critical points are $(0, 0)$, $(0, 5/3)$, $(1, 0)$ and $(2, 2)$.

(c) The Jacobian matrix is

$$\mathbf{J} = \begin{pmatrix} 1 - 2x + 0.5y & 0.5x \\ 0.25y & 2.5 - 3y + 0.25x \end{pmatrix}.$$

At $(0, 0)$,

$$\mathbf{J}(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 2.5 \end{pmatrix}.$$

The associated eigenvalues and eigenvectors are $r_1 = 1$, $\boldsymbol{\xi}_1 = (1, 0)^T$ and $r_2 = 2.5$, $\boldsymbol{\xi}_2 = (0, 1)^T$. The eigenvalues are positive. Therefore, the origin is an unstable node.
At $(0, 5/3)$,

$$\mathbf{J}(0, 5/3) = \begin{pmatrix} 11/6 & 0 \\ 5/12 & -5/2 \end{pmatrix}.$$

The associated eigenvalues and eigenvectors are $r_1 = 11/6$, $\boldsymbol{\xi}_1 = (52/5, 1)^T$ and $r_2 = -5/2$, $\boldsymbol{\xi}_2 = (0, 1)^T$. The eigenvalues are of opposite sign. Therefore, $(0, 5/3)$ is a saddle, which is unstable.
At $(1, 0)$,

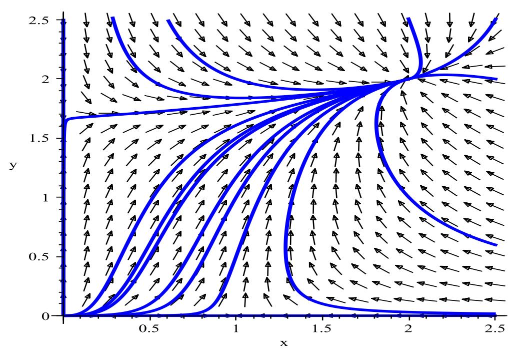
$$\mathbf{J}(1, 0) = \begin{pmatrix} -1 & 1/2 \\ 0 & 11/4 \end{pmatrix}.$$

The associated eigenvalues and eigenvectors are $r_1 = -1$, $\boldsymbol{\xi}_1 = (1, 0)^T$ and $r_2 = 11/4$, $\boldsymbol{\xi}_2 = (1, 15/2)^T$. The eigenvalues are of opposite sign. Therefore, this critical point is a saddle, which is unstable.
At $(2, 2)$,

$$\mathbf{J}(2, 2) = \begin{pmatrix} -2 & 1 \\ 1/2 & -3 \end{pmatrix}.$$

The associated eigenvalues and eigenvectors are $r_1 = (-5+\sqrt{3})/2$, $\boldsymbol{\xi}_1 = (1, (-1+\sqrt{3})/2)^T$ and $r_2 = -(5+\sqrt{3})/2$, $\boldsymbol{\xi}_2 = (1, -(1+\sqrt{3})/2)^T$. The eigenvalues are both negative. Therefore, this critical point is a stable node, which is asymptotically stable.

(d,e)



(f) Nonzero solutions converge toward the equilibrium point $(2, 2)$.

8.(a) The critical points are solutions of

$$x(\epsilon_1 - \sigma_1 x - \alpha_1 y) = 0$$
$$y(\epsilon_2 - \sigma_2 y - \alpha_2 x) = 0.$$

If $x = 0$, then either $y = 0$ or $y = \epsilon_2/\sigma_2$. If $y = 0$, then either $x = 0$ or $x = \epsilon_1/\sigma_1$. The fourth critical point is given by

$$\left(\frac{\epsilon_1\sigma_2 - \epsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2}, \frac{\epsilon_2\sigma_1 - \epsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2}\right).$$

If $\epsilon_2/\alpha_2 > \epsilon_1/\sigma_1$ and $\epsilon_2/\sigma_2 > \epsilon_1/\alpha_1$, then $\epsilon_2\sigma_1 - \epsilon_1\alpha_2 > 0$ and $\epsilon_1\sigma_2 - \epsilon_2\alpha_1 < 0$, thus either the x or the y coordinate of the last critical point is negative so both species cannot survive. The Jacobian matrix is

$$\mathbf{J} = \begin{pmatrix} \epsilon_1 - 2\sigma_1 x - \alpha_1 y & -\alpha_1 x \\ -\alpha_2 y & \epsilon_2 - 2\sigma_2 y - \alpha_2 x \end{pmatrix}.$$

At $(0, 0)$,

$$\mathbf{J}(0,0) = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}.$$

The associated eigenvalues are $r = \epsilon_1, \epsilon_2$. Since both eigenvalues are positive, the origin is an unstable node. At $(0, \epsilon_2/\sigma_2)$,

$$\mathbf{J}(0, \epsilon_2/\sigma_2) = \begin{pmatrix} \epsilon_1 - \alpha_1\epsilon_2/\sigma_2 & 0 \\ -\epsilon_2\alpha_2/\sigma_2 & -\epsilon_2 \end{pmatrix}.$$

The associated eigenvalues are $r = \epsilon_1 - \alpha_1\epsilon_2/\sigma_2, -\epsilon_2$. Here $\alpha_1\epsilon_2 - \epsilon_1\sigma_2 > 0$, so both eigenvalues are negative, and the point $(0, \epsilon_2/\sigma_2)$ is a stable node, which is asymptotically stable. At $(\epsilon_1/\sigma_1, 0)$,

$$\mathbf{J}(\epsilon_1/\sigma_1, 0) = \begin{pmatrix} -\epsilon_1 & -\epsilon_1\alpha_1/\sigma_1 \\ 0 & (\sigma_1\epsilon_2 - \epsilon_1\alpha_2)/\sigma_1 \end{pmatrix}.$$

The associated eigenvalues are $r = (\sigma_1\epsilon_2 - \epsilon_1\alpha_2)/\sigma_1,\ -\epsilon_1$. Here $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 > 0$, so the eigenvalues are of opposite sign, and the point $(\epsilon_1/\sigma_1, 0)$ is a saddle, which is unstable. Thus the fish represented by y (redear) survive.

(b) The analysis here is similar to part (a); again, one of the coordinates of the fourth equilibrium point is negative, and hence a mixed state is not possible. The stability analysis shows that the bluegill (represented by x) survive in this case.

9.(a) We compute:

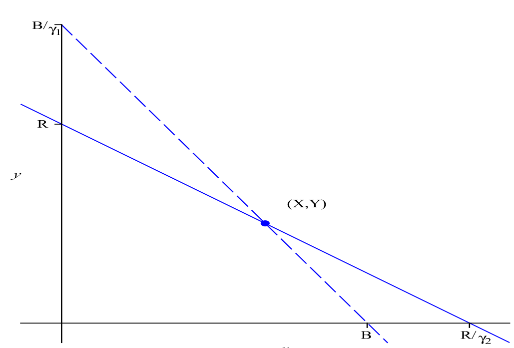
$$x' = \epsilon_1 x(1 - \frac{\sigma_1}{\epsilon_1}x - \frac{\alpha_1}{\epsilon_1}y) = \epsilon_1 x(1 - \frac{1}{B}x - \frac{\gamma_1}{B}y)$$

and

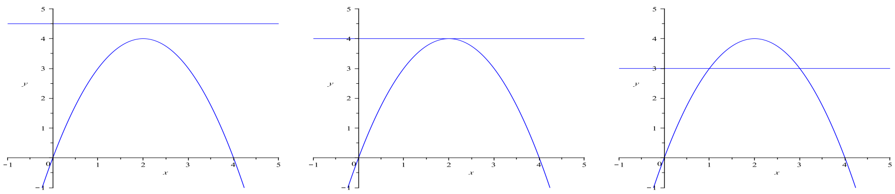
$$y' = \epsilon_2 y(1 - \frac{\sigma_2}{\epsilon_2}y - \frac{\alpha_2}{\epsilon_2}x) = \epsilon_2 y(1 - \frac{1}{R}y - \frac{\gamma_2}{R}x).$$

The coexistence equilibrium point is given by $x + \gamma_1 y = B$ and $\gamma_2 x + y = R$. Solving these yields $X = (B - \gamma_1 R)/(1 - \gamma_1\gamma_2)$ and $Y = (R - \gamma_2 B)/(1 - \gamma_1\gamma_2)$.

(b) If B is reduced, it is clear from the answer to part (a) that X is reduced and Y is increased. To determine whether the bluegill will die out, we give an intuitive argument which can be confirmed by doing the analysis. Note that $B/\gamma_1 = \epsilon_1/\alpha_1 > \epsilon_2/\sigma_2 = R$ and $R/\gamma_2 = \epsilon_2/\alpha_2 > \epsilon_1/\sigma_1 = B$ so that the graph of the lines $1 - x/B - \gamma_1 y/B = 0$ and $1 - y/R - \gamma_2 x/R = 0$ must intersect in the first quadrant. As B is decreased, X decreases, Y increases and the point of intersection moves closer to $(0, R)$. If $B/\gamma_1 < R$, coexistence is not possible, and the only critical points are $(0, 0)$, $(0, R)$ and $(B, 0)$. It can be shown that $(0, 0)$ and $(B, 0)$ are unstable and $(R, 0)$ is asymptotically stable. Hence we conclude, when coexistence is no longer possible, that $x \to 0$ and $y \to R$ and thus the bluegill population will die out.



13.(a) Nullclines:



(a) $\alpha = 3$ (b) $\alpha = 8/3$ (c) $\alpha = 2$

(b) The critical points are solutions of

$$-4x + y + x^2 = 0$$
$$\frac{3}{2}\alpha - y = 0.$$

The solutions of these equations are

$$\left(2 \pm \sqrt{4 - \frac{3}{2}\alpha}, \frac{3}{2}\alpha\right)$$

and exist for $\alpha \leq 8/3$.

(c) For $\alpha = 2$, the critical points are $(1, 3)$ and $(3, 3)$. The Jacobian matrix is

$$\mathbf{J} = \begin{pmatrix} -4 + 2x & 1 \\ 0 & -1 \end{pmatrix}.$$

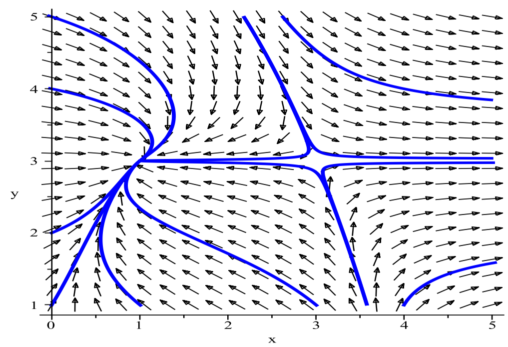
At $(1, 3)$,

$$\mathbf{J}(1, 3) = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues are $r = -2, -1$. Since they are both negative, $(1, 3)$ is a stable node, which is asymptotically stable.
At $(3, 3)$,

$$\mathbf{J}(3, 3) = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}.$$

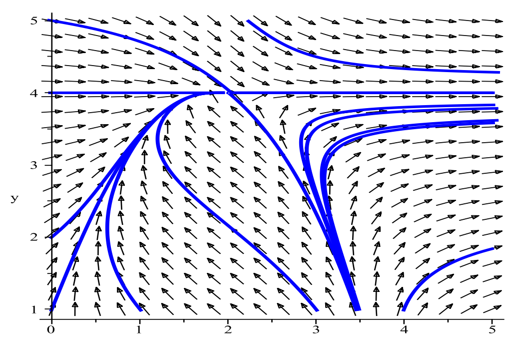
The eigenvalues are $r = 2, -1$. Since they are of opposite sign $(3, 3)$ is a saddle, which is unstable.



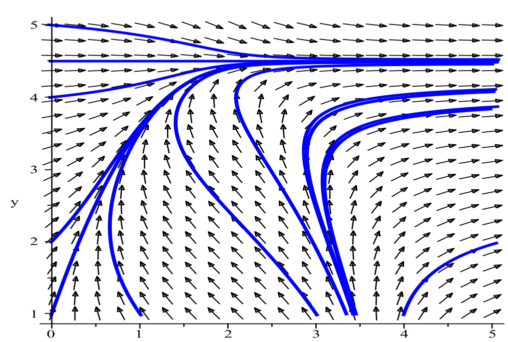
(d) From part (a), the bifurcation value is $\alpha_0 = 8/3$. At this value α_0, the critical point is $(2, 4)$. The Jacobian matrix is

$$\mathbf{J}(2,4) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

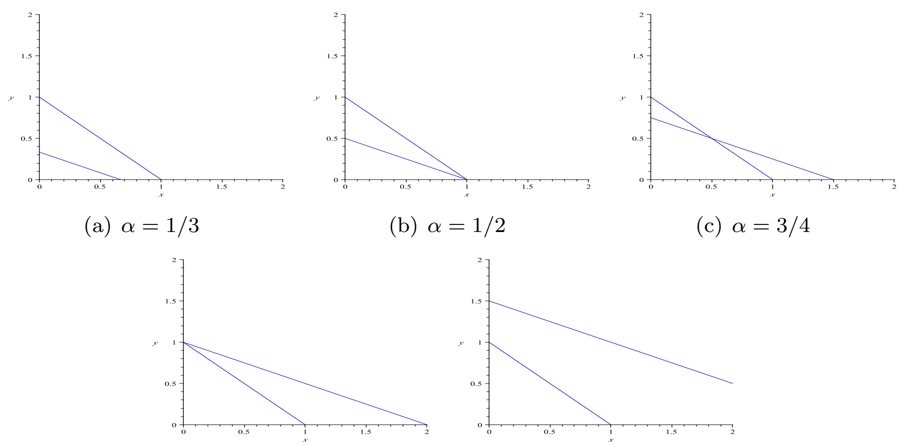
The eigenvalues are $r = 0, -1$.



(f) Below we show the phase portrait for $\alpha = 3$, which has no critical points.



17.(a) Nullclines:



(a) $\alpha = 1/3$ (b) $\alpha = 1/2$ (c) $\alpha = 3/4$

(d) $\alpha = 1$ (e) $\alpha = 3/2$

(b) The equilibrium points are $P_1(0,0)$, $P_2(1,0)$, $P_3(0,\alpha)$ and $P_4(2-2\alpha, 2\alpha-1)$. The fourth equilibrium point is in the first quadrant as long as $1/2 \le \alpha \le 1$.

(c) When $\alpha = 0$, P_3 coincides with P_1, when $\alpha = 0.5$, P_4 coincides with P_2, and when $\alpha = 1$, P_4 coincides with P_3. These are the bifurcation points, since in each case two of the four points have the same coordinates.

(d,e) The Jacobian is

$$\mathbf{J} = \begin{pmatrix} 1-2x-y & -x \\ -y/2 & \alpha - 2y - x/2 \end{pmatrix}.$$

This means that at the origin

$$\mathbf{J}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix},$$

so the origin is an unstable node when $\alpha > 0$. (The eigenvalues are 1 and α.)
At the critical point $(1,0)$ the Jacobian is

$$\mathbf{J}(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & \alpha - 1/2 \end{pmatrix},$$

which means that this equilibrium is a saddle when $\alpha > 1/2$ and an asymptotically stable node when $0 < \alpha < 1/2$.
At the critical point $(0,\alpha)$ the Jacobian is

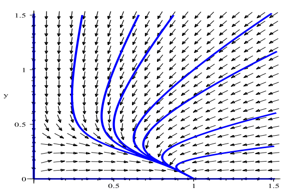
$$\mathbf{J}(0,\alpha) = \begin{pmatrix} 1-\alpha & 0 \\ -\alpha/2 & -\alpha \end{pmatrix},$$

which implies that this equilibrium is a saddle when $0 < \alpha < 1$ and an asymptotically stable node when $\alpha > 1$.
At the critical point $(2(1-\alpha), 2\alpha - 1)$ the Jacobian is given by

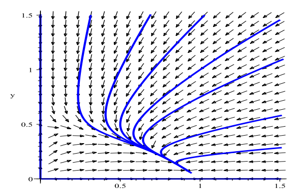
$$\mathbf{J}(2(1-\alpha), 2\alpha - 1) = \begin{pmatrix} 2(\alpha - 1) & 2(\alpha - 1) \\ 1/2 - \alpha & 1 - 2\alpha \end{pmatrix}.$$

It can be shown that this is an asymptotically stable node when $1/2 < \alpha < 1$.
At the bifurcation value $\alpha = 0$, the origin changes from a saddle point to an unstable node, while $(0, \alpha)$ changes from an unstable node to a saddle point. At $\alpha = 0.5$, $(1, 0)$ changes from an asymptotically stable node to a saddle point, while P_4 changes from a saddle point to an asymptotically stable node. At $\alpha = 1$, $(0, \alpha)$ changes from a saddle point to an asymptotically stable node, while P_4 changes from an asymptotically stable node to a saddle point.

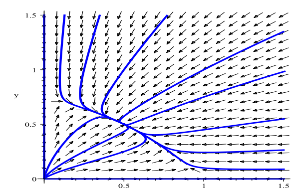
(f) Phase portraits:



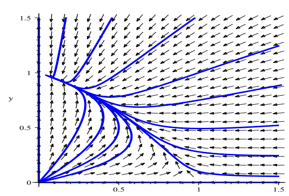
(a) $\alpha = 1/3$



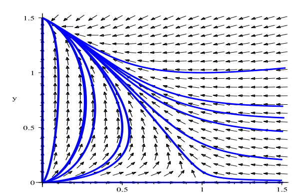
(b) $\alpha = 1/2$



(c) $\alpha = 3/4$



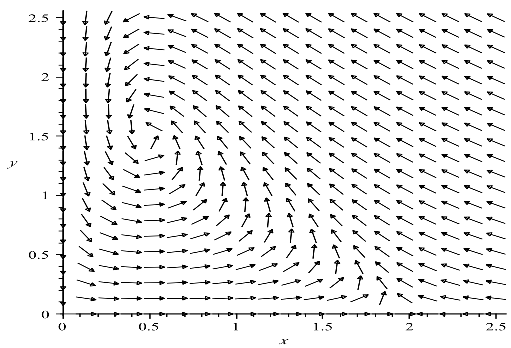
(d) $\alpha = 1$



(e) $\alpha = 3/2$

9.5

3.(a)



(b) The critical points are solutions of the system

$$x(1-0.5x-0.5y)=0$$
$$y(-0.25+0.5x)=0.$$

The three critical points are $(0,0)$, $(2,0)$ and $(1/2,3/2)$.

(c) The Jacobian matrix is

$$\mathbf{J}=\begin{pmatrix}1-x-y/2 & -x/2\\ y/2 & -1/4+x/2\end{pmatrix}.$$

At $(0,0)$,

$$\mathbf{J}(0,0)=\begin{pmatrix}1 & 0\\ 0 & -1/4\end{pmatrix}.$$

The eigenvalues and eigenvectors are $r_1=1$, $\boldsymbol{\xi}_1=(1,0)^T$ and $r_2=-1/4$, $\boldsymbol{\xi}_2=(0,1)^T$. The eigenvalues are of opposite sign. Therefore, $(0,0)$ is a saddle point, which is unstable.
At $(2,0)$,

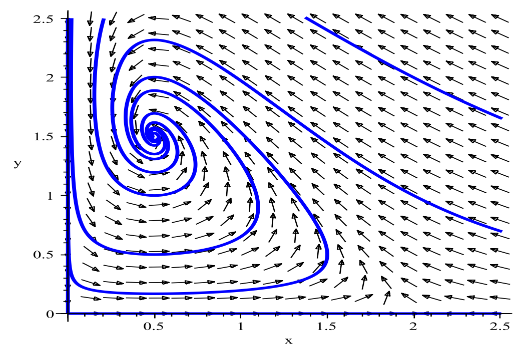
$$\mathbf{J}(2,0)=\begin{pmatrix}-1 & -1\\ 0 & 3/4\end{pmatrix}.$$

The eigenvalues and eigenvectors are $r_1=-1$, $\boldsymbol{\xi}_1=(1,0)^T$ and $r_2=3/4$, $\boldsymbol{\xi}_2=(1,-7/4)^T$. The eigenvalues have opposite sign. Therefore, $(2,0)$ is a saddle point, which is unstable.
At $(1/2,3/2)$,

$$\mathbf{J}(1/2,3/2)=\begin{pmatrix}-1/4 & -1/4\\ 3/4 & 0\end{pmatrix}.$$

The eigenvalues and eigenvectors are $r_1=(-1+i\sqrt{11})/8$, $\boldsymbol{\xi}_1=((-1+i\sqrt{11})/6,1)^T$ and $r_2=(-1-i\sqrt{11})/8$, $\boldsymbol{\xi}_2=((-1-i\sqrt{11})/6,1)^T$. The eigenvalues have negative real part. Therefore, $(1/2,3/2)$ is a stable spiral, which is asymptotically stable.
(As in Section 9.4, the Jacobian, evaluated at each of the critical points, can be easily used to find the associated linear system at each critical point.)

(d,e)



(f) Except for solutions along the coordinate axes, the other trajectories spiral towards the critical point $(1/2, 3/2)$.

7.(a) Looking at the coefficient of the trigonometric functions in equations (24), we see that the ratio will be given by

$$\frac{(cK/\gamma)}{(a/\alpha)\sqrt{c/aK}} = \frac{\alpha\sqrt{c}}{\gamma\sqrt{a}}.$$

(b) For system (2), $a = 1$, $\alpha = 0.5$, $c = 0.75$ and $\gamma = 0.25$. Therefore, the ratio is $0.5\sqrt{0.75}/0.25 = \sqrt{3}$.

(c) The amplitude for the prey function in Figure 9.5.3 is approximately 2.5 and the amplitude for the predator function in Figure 9.5.3 is approximately 1.45. Using these approximations, we say that the ratio is approximately 1.72 which is close to $\sqrt{3}$. In this case the linear approximation is a good predictor.

11.(a) Looking at the equation for $x' = 0$, we need $x = 0$ or $\sigma x + 0.5y = 1$. Looking at the equation for $y' = 0$, we need $y = 0$ or $x = 3$. Therefore, the critical points are given by $(0, 0)$, $(1/\sigma, 0)$ and $(3, 2 - 6\sigma)$. As σ increases from zero, the critical point $(1/\sigma, 0)$ approaches the origin, and the critical point $(3, 2 - 6\sigma)$ will eventually leave the first quadrant and enter the fourth quadrant. At $\sigma = 1/3$ they will coincide.

(b) Here, we have $F(x, y) = x(1 - \sigma x - 0.5y)$ and $G(x, y) = y(-0.75 + 0.25x)$. Therefore, the Jacobian matrix for this system is

$$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 1 - 2\sigma x - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix}.$$

We will look at the linear systems near the critical points above. At the critical point $(0, 0)$, the Jacobian matrix is

$$\begin{pmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3/4 \end{pmatrix}$$

and the corresponding linear system near $(0,0)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3/4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

Near the critical point $(1/\sigma, 0)$, the Jacobian matrix is

$$\begin{pmatrix} F_x(1/\sigma,0) & F_y(1/\sigma,0) \\ G_x(1/\sigma,0) & G_y(1/\sigma,0) \end{pmatrix} = \begin{pmatrix} -1 & -1/2\sigma \\ 0 & -3/4+1/4\sigma \end{pmatrix}$$

and the corresponding linear system near $(1/\sigma, 0)$ is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & -1/2\sigma \\ 0 & -3/4+1/4\sigma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - 1/\sigma$ and $v = y$. Near the critical point $(3, 2-6\sigma)$, the Jacobian matrix is

$$\begin{pmatrix} F_x(3,2-6\sigma) & F_y(3,2-6\sigma) \\ G_x(3,2-6\sigma) & G_y(3,2-6\sigma) \end{pmatrix} = \begin{pmatrix} -3\sigma & -3/2 \\ 1/2-3\sigma/2 & 0 \end{pmatrix}$$

and the corresponding linear system near $(3, 2-6\sigma)$ is

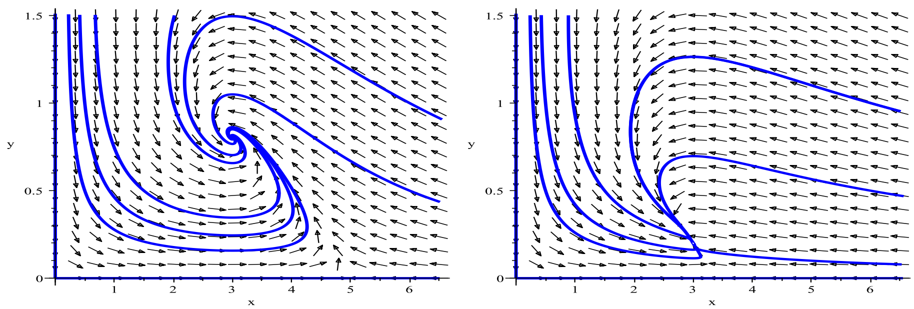
$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -3\sigma & -3/2 \\ 1/2-3\sigma/2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - 3$ and $v = y - 2 + 6\sigma$.

The eigenvalues for the linearized system near $(0,0)$ are given by $r = 1, -3/4$. Therefore, $(0,0)$ is a saddle point. The eigenvalues for the linearized system near $(1/\sigma, 0)$ are given by $r = -1, (1-3\sigma)/(4\sigma)$. For $\sigma < 1/3$, there will be one positive eigenvalue and one negative eigenvalue. In this case, $(1/\sigma, 0)$ will be a saddle point. For $\sigma > 1/3$, both eigenvalues will be negative, in which case $(1/\sigma, 0)$ will be an asymptotically stable node. The eigenvalues for the linearized system near $(3, 2-6\sigma)$ are $r = (-3\sigma \pm \sqrt{9\sigma^2 + 9\sigma - 3})/2$. Solving the polynomial equation $9\sigma^2 + 9\sigma - 3 = 0$, we see that the eigenvalues will have non-zero imaginary part if $0 < \sigma < (\sqrt{21} - 3)/6$. In this case, since the real part, -3σ will be negative, the point $(3, 2-6\sigma)$ will be an asymptotically stable spiral point. If $\sigma > (\sqrt{21} - 3)/6$, then the eigenvalues will both be real. We just need to determine whether they will have the same sign or opposite signs. Solving the equation $-3\sigma + \sqrt{9\sigma^2 + 9\sigma - 3} = 0$, we see that the cut-off is $\sigma = 1/3$. In particular, we conclude that if $(\sqrt{21} - 3)/6 < \sigma < 1/3$, then this critical point will have two real-valued eigenvalues which are negative, in which case this critical point will be an asymptotically stable node. If $\sigma > 1/3$, however, the eigenvalues will be real-valued, but with opposite signs, in which case $(3, 2-6\sigma)$ will be a saddle point.

We see that the critical point $(3, 2-6\sigma)$ is the critical point in the first quadrant if $0 < \sigma < 1/3$. From the analysis above, we see that the nature of the critical point changes at $\sigma_1 = (\sqrt{21} - 3)/6$. In particular, at this value of σ, the critical point switches from an asymptotically stable spiral point to an asymptotically stable node.

(c) The two phase portraits below are shown $\sigma = 0.2$ and $\sigma = 0.3$, respectively.



(d) As σ increases, the spiral behavior disappears. For smaller values of σ, the number of prey will decrease, causing a decrease in the number of predators, but then triggering an increase in the number of prey and eventually an increase in the number of predators. This cycle will continue to repeat as the system approaches the equilibrium point. As the value for σ increases, the cycling behavior between the predators and prey goes away.

14.(a) If the prey are harvested, then there will be less prey available for the predators, thus causing a decrease in the number of predators and allowing more of the prey that are not harvested to survive. (As we will see below, the number of prey will not change.) If the predators are harvested, then there will be less predators to eat the prey, thus, allowing the number of prey to increase. As a result, with more prey available, a larger percentage of the predators which are not harvested will be able to survive. (As we will see below the total number of predators will remain the same.) If they are both harvested, then initially there will be less prey available for the predators, causing a decrease in the number of predators, thus leading to an increase in the number of prey which survive.

(b) The equilibrium solution will occur when $x' = 0$, $y' = 0$. Solving these equations, we see that the equilibrium solution (with non-zero amounts of predators and prey) is given by $((c + E_2)/\gamma, (a - E_1)/\alpha)$. Therefore, if $E_1 > 0$ and $E_2 = 0$, then the number of prey stay the same, but the number of predators decreases.

(c) Using the equilibrium solution from part (b), we see that if $E_1 = 0$, $E_2 > 0$, then the number of prey increases, while the number of predators stays the same.

(d) If both $E_1 > 0$ and $E_2 > 0$, then the number of prey increases and the number of predators decreases.

9.6

1. We consider the function $V(x, y) = a\,x^2 + c\,y^2$. The rate of change of V along any trajectory is

$$\dot{V} = V_x \frac{dx}{dt} + V_y \frac{dy}{dt} = 2ax(-x^3 + xy^2) + 2cy(-2x^2y - y^3) =$$

$$= -(2ax^4 + 2(2c - a)x^2y^2 + 2cy^4).$$

If we choose a and c be any positive real numbers with $2c \geq a$, then $\dot{V}(x, y)$ is negative definite. By definition V is positive definite. It follows from Theorem 9.6.1 that the origin is an asymptotically stable critical point.

3. We consider the function $V(x, y) = a\,x^2 + c\,y^2$. The rate of change of V along any trajectory is

$$\dot{V} = V_x \frac{dx}{dt} + V_y \frac{dy}{dt} = 2ax(-x^3 + 2y^3) + 2cy(-2xy^2) = -2ax^4 + 4(a - c)xy^3.$$

If we choose a and c to be any positive real numbers with $a = c$, then $\dot{V}(x, y) = -2ax^4 \leq 0$ in any neighborhood containing the origin and thus $\dot{V}$ is negative semidefinite. By definition V is positive definite. It follows from Theorem 9.6.1 that the origin is a stable critical point. However, the origin may still be asymptotically stable although the $V(x, y)$ used here is not sufficient to prove that.

6.(a) The system is $dx/dt = y$ and $dy/dt = -g(x)$. Since $g(0) = 0$, we conclude that $(0, 0)$ is a critical point.

(b) From the given conditions, the graph of g must be positive for $0 < x < k$ and negative for $-k < x < 0$. Thus if $0 < x < k$, then $\int_0^x g(s)\,ds > 0$, and if $-k < x < 0$, then $\int_0^x g(s)\,ds = -\int_x^0 g(s)\,ds > 0$. Since $V(0, 0) = 0$ it follows that $V(x, y) = y^2/2 + \int_0^x g(s)\,ds$ is positive definite for $-k < x < k$, $-\infty < y < \infty$. Next, we have $\dot{V}(x, y) = V_x(dx/dt) + V_y(dy/dt) = g(x)y + y(-g(x)) = 0$. Theorem 9.6.1 shows that the origin is at least a stable equilibrium point.

7.(a) The right side of both equations is zero at the origin, so it is a critical point.

(b) V is positive definite by Theorem 9.6.4. Since $V_x = 2x$ and $V_y = 2y$, we get $\dot{V}(x, y) = V_x(dx/dt) + V_y(dy/dt) = 2xy - 2y^2 - 2y \sin x = 2y(-y + x - \sin x)$. Since $x - \sin x < 0$ for $x < 0$ we have $\dot{V}(x, y) < 0$ for all $y > 0$. If $x > 0$, choose y so that $0 < y < x - \sin x$. Then $\dot{V}(x, y) > 0$. Hence V is not a Liapunov function.

(c) Since $V(0, 0) = 0$, $1 - \cos x > 0$ for $0 < |x| < 2\pi$ and $y^2 > 0$ for $y \neq 0$, it follows that $V(x, y)$ is positive definite in a neighborhood of the origin. Next $V_x(x, y) = \sin x$, $V_y(x, y) = y$, so $\dot{V}(x, y) = V_x(dx/dt) + V_y(dy/dt) = (\sin x)y + y(-y - \sin x) = -y^2$. Hence $\dot{V}$ is negative semidefinite and $(0, 0)$ is a stable critical point by Theorem 9.6.1.

(d) $V(x, y) = (x + y)^2/2 + x^2 + y^2/2 = 3x^2/2 + xy + y^2$ is positive definite by Theorem 9.6.4. Next $V_x = 3x + y$ and $V_y = x + 2y$, so

$$\dot{V}(x, y) = (3x + y)y - (x + 2y)(y + \sin x) = 2xy - y^2 - (x + 2y)\sin x =$$

$$= 2xy - y^2 - (x + 2y)(x - \alpha x^3/6) \text{ [from the hint]}$$

$$= -x^2 - y^2 + \alpha(x + 2y)x^3/6 = -r^2 + \alpha r^4(\cos\theta + 2\sin\theta)\cos^3\theta/6 <$$

$$< -r^2 + r^4/2 = -r^2(1 - r^2/2).$$

Thus $\dot{V}$ is negative definite for $r < \sqrt{2}$ and from Theorem 9.6.1 it follows that the origin is an asymptotically stable critical point.

8. Let $x = u$ and $y = du/dt$ to obtain the system $dx/dt = y$ and $dy/dt = -c(x)y - g(x)$. Now consider $V(x, y) = y^2/2 + \int_0^x g(s)\,ds$, which yields $\dot{V} = g(x)y + y(-c(x)y - g(x)) = -y^2 c(x)$, which is negative semidefinite and we obtain that the origin is stable by Theorem 9.6.1.

10.(a) For asymptotic stability we need that the real parts of all the eigenvalues of the coefficient matrix are negative. Looking at all the possible cases (see Problem 21 in Section 9.1) we obtain that this happens if an only if $a_{11} + a_{22} < 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0$.

(b) Since $V_x = 2Ax + By$, $v_y = Bx + 2Cy$, we have

$$\dot{V} = (2Ax + By)(a_{11}x + a_{12}y) + (Bx + 2Cy)(a_{21}x + a_{22}y) =$$

$$= (2Aa_{11} + Ba_{21})x^2 + (2(Aa_{12} + Ca_{21}) + B(a_{11} + a_{22}))xy + (2Ca_{22} + Ba_{12})y^2.$$

We choose A, B and C so that $2Aa_{11} + Ba_{21} = -1$, $2(Aa_{12} + Ca_{21}) + B(a_{11} + a_{22}) = 0$ and $2Ca_{22} + Ba_{12} = -1$. The first and third equations give us A and C in terms of B, respectively. We substitute in the second equation to find B and then calculate A and C. The result is given in the text.

(c) Since $a_{11}a_{22} - a_{12}a_{21} > 0$ and $a_{11} + a_{22} < 0$, we see that $\Delta < 0$ and so $A > 0$. Using the expression for A, B, and C found in part (b) we obtain

$$(4AC - B^2)\Delta = (a_{21}^2 + a_{22}^2 + (a_{11}a_{22} - a_{12}a_{21}))(a_{11}^2 + a_{12}^2 + (a_{11}a_{22} - a_{12}a_{21}))$$

$$-(a_{12}a_{22} + a_{11}a_{21})^2 = (a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2)(a_{11}a_{22} - a_{12}a_{21}) +$$

$$(a_{11}^2 + a_{12}^2)(a_{21}^2 + a_{22}^2) + (a_{11}a_{22} - a_{12}a_{21})^2 - (a_{12}a_{22} + a_{11}a_{21})^2 =$$

$$(a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2)(a_{11}a_{22} - a_{12}a_{21}) + 2(a_{11}a_{22} - a_{12}a_{21})^2.$$

Since $a_{11}a_{22} - a_{12}a_{21} > 0$ it follows that $4AC - B^2 > 0$.

11.(a) For $V(x, y) = Ax^2 + Bxy + Cy^2$ we have

$$\dot{V} = (2Ax + By)(a_{11}x + a_{12}y + F_1(x, y)) + (Bx + 2Cy)(a_{21}x + a_{22}y + G_1(x, y)) =$$

$$(2Ax + By)(a_{11}x + a_{12}y) + (Bx + 2Cy)(a_{21}x + a_{22}y) + (2Ax + By)F_1(x, y) +$$

$$+(Bx + 2Cy)G_1(x, y) = -x^2 - y^2 + (2Ax + By)F_1(x, y) + (Bx + 2Cy)G_1(x, y),$$

if A, B, and C are chosen as in Problem 10.

(b) Substituting $x = r\cos\theta$, $y = r\sin\theta$ we find that

$$\dot{V} = -r^2 + r(2A\cos\theta + B\sin\theta)F_1(r\cos\theta, r\sin\theta) +$$

$$+r(B\cos\theta + 2C\sin\theta)G_1(r\cos\theta, r\sin\theta).$$

Now we make use of the facts that there exists an M such that $|2A| \le M$, $|B| \le M$, and $|2C| \le M$ and that given any $\varepsilon > 0$ there exists a circle with radius R such that

$|F_1(x,y)| < \varepsilon r$ and $|G_1(x,y)| < \varepsilon r$ for $0 < r < R$. We have $|2A\cos\theta + B\sin\theta| \le 2M$ and $|B\cos\theta + 2C\sin\theta| \le 2M$. Thus $\dot{V} \le -r^2 + 2Mr(\varepsilon r) + 2Mr(\varepsilon r) = -r^2(1 - 4M\varepsilon)$. If we choose $\varepsilon = M/8$ we obtain $\dot{V} \le -r^2/2$ for $0 \le r < R$. Hence $\dot{V}$ is negative definite in $0 \le r < R$ and from Problem 10(c) V is positive definite and thus V is a Liapunov function for the almost linear system.

9.7

1. Note that $r = 1$, $\theta = t + t_0$ satisfy the two equations for all t and is thus a periodic solution. We notice that for $0 < r < 1$, $dr/dt > 0$, while for $r > 1$, $dr/dt < 0$. Therefore, $r = 0$ is an unstable critical point, while $r = 1$ is an asymptotically stable critical point (for the dr/dt equation). Thus a limit cycle is given by $r = 1$, $\theta = t + t_0$, which is asymptotically stable.

2. $r = 1$, $\theta = -t + t_0$ is a periodic solution. We notice that for $0 < r < 1$, $dr/dt > 0$, while for $r > 1$, $dr/dt > 0$, as well. Therefore, $r = 0$ is an unstable critical point, while $r = 1$ is a semistable critical point (for the dr/dt equation). Thus a limit cycle is given by $r = 1$, $\theta = -t + t_0$, which is semistable.

4. $r = 1$, $\theta = -t + t_0$ and $r = 2$, $\theta = -t + t_0$ are periodic solutions. We notice that $dr/dt < 0$ for $0 < r < 1$ and $r > 2$, while $dr/dt > 0$ for $1 < r < 2$. Therefore, $r = 0$ is a stable critical point, $r = 1$ is an unstable critical point, and $r = 2$ is an asymptotically stable critical point (for the dr/dt equation). Thus a limit cycle is given by $r = 1$, $\theta = -t + t_0$, which is unstable. Another limit cycle is given by $r = 2$, $\theta = -t + t_0$, which is asymptotically stable.

7. We compute:

$$y\frac{dx}{dt} - x\frac{dy}{dt} = r\sin\theta\left[\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right] - r\cos\theta\left[\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right] =$$

$$= -r^2\sin^2\theta\frac{d\theta}{dt} - r^2\cos^2\theta\frac{d\theta}{dt} = -r^2\frac{d\theta}{dt}.$$

8.(a) Since $r^2 = x^2 + y^2$, we have

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

Thus

$$r\frac{dr}{dt} = \frac{x^2 f(r)}{r} + \frac{y^2 f(r)}{r} = \frac{r^2 f(r)}{r} = rf(r).$$

Therefore, $rdr/dt = rf(r)$, which implies $dr/dt = f(r)$. Therefore, we have periodic solutions corresponding to the zeros of $f(r)$. To find the direction of motion on the closed trajectories, we use the result from Problem 7:

$$-r^2\frac{d\theta}{dt} = y\frac{dx}{dt} - x\frac{dy}{dt}.$$

Therefore, for this system, we have

$$-r^2 \frac{d\theta}{dt} = -y^2 + \frac{xyf(r)}{r} - x^2 - \frac{xyf(r)}{r} = -x^2 - y^2 = -r^2.$$

Therefore, $d\theta/dt = 1$, which implies $\theta = t + t_0$. Therefore, the closed trajectories will move in the counter-clockwise direction.

(b) By part (a), we know the periodic solutions will be given by the zeros of f. The zeros are $r = 0, 1, 2, 3$. Using the fact that $dr/dt = f(r)$, we see that $dr/dt > 0$ if $0 < r < 1$ and $r > 3$ and $dr/dt < 0$ if $1 < r < 2$ and $2 < r < 3$. Therefore, $r = 0$ is unstable, $r = 1$ is asymptotically stable, $r = 2$ is semistable, and $r = 3$ is unstable. We conclude that there is an asymptotically stable limit cycle at $r = 1$ with $\theta = t + t_0$, a semistable limit cycle at $r = 2$ with $\theta = t + t_0$ and an unstable periodic solution at $r = 3$ with $\theta = t + t_0$.

9. Using the fact that

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt},$$

we have

$$r\frac{dr}{dt} = xy + \frac{x^2}{\sqrt{x^2+y^2}}(x^2+y^2-2) - xy + \frac{y^2}{\sqrt{x^2+y^2}}(x^2+y^2-2) = r(r^2-2).$$

Therefore, $dr/dt = r^2 - 2$. Therefore, we have one critical point at $r = \sqrt{2}$. We see that $dr/dt > 0$ if $r > \sqrt{2}$ and $dr/dt < 0$ if $r < \sqrt{2}$. Therefore, $r = \sqrt{2}$ is an unstable periodic solution. To find the direction of motion on the closed trajectories, we use the fact that

$$-r^2 \frac{d\theta}{dt} = y\frac{dx}{dt} - x\frac{dy}{dt}.$$

Therefore, here we have

$$-r^2 \frac{d\theta}{dt} = y^2 + x^2 = r^2.$$

Therefore, $d\theta/dt = -1$, which implies $\theta = -t + t_0$.

11. Given that $F(x, y) = x + y + x^3 - y^2$ and $G(x, y) = -x + 2y + x^2 y + y^3/3$,

$$F_x + G_y = 3 + 4x^2 + y^2$$

is positive for all (x, y). Therefore, by Theorem 9.7.2, the system cannot have a non-constant periodic solution.

13. We parametrize the curve C by t. Therefore, we can rewrite the line integral as

$$\begin{aligned}\int_C [F(x,y)\,dy - G(x,y)\,dx] &= \int_t^{t+T} [F(\phi(t),\psi(t))\psi'(t) - G(\phi(t),\psi(t))\phi'(t)]\,dt \\ &= \int_t^{t+T} [\phi'(t)\psi'(t) - \psi'(t)\phi'(t)]\,dt = 0.\end{aligned}$$

Then, using Green's Theorem, we must have

$$\iint_R [F_x(x,y) + G_y(x,y)]\, dA = 0.$$

If $F_x + G_y$ has the same sign throughout D, then the double integral cannot be zero. Thus either the solution of Eqs.(15) is not periodic or if it is, it cannot lie entirely in D.

18.(a) The critical points must satisfy $x = 0$ or $0.2x + 2y/(x+6) = 2.4$ and $y = 0$ or $x/(x+6) = 0.25$. Solving these equations, we see that the critical points are $(0,0)$, $(12,0)$ and $(2,8)$.

(b) To determine the type and stability of each critical point, we need to look at the Jacobian matrix

$$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 2.4 - 0.4x - \dfrac{12y}{(x+6)^2} & -\dfrac{2x}{x+6} \\ \dfrac{6y}{(x+6)^2} & -0.25 + \dfrac{x}{x+6} \end{pmatrix}.$$

Therefore, the Jacobian near $(0,0)$ is

$$\begin{pmatrix} 2.4 & 0 \\ 0 & -0.25 \end{pmatrix}.$$

The Jacobian matrix near $(12,0)$ is

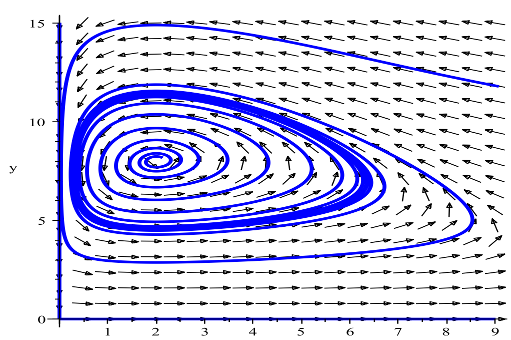
$$\begin{pmatrix} -12/5 & -4/3 \\ 0 & 5/12 \end{pmatrix}.$$

The Jacobian matrix near $(2,8)$ is

$$\begin{pmatrix} 1/10 & -1/2 \\ 3/4 & 0 \end{pmatrix}.$$

The eigenvalues for $(0,0)$ are $r = 2.4, -0.25$. Therefore, $(0,0)$ is a saddle point. The eigenvalues for $(12,0)$ are $r = -12/5, 5/12$. Therefore, $(12,0)$ is a saddle point. The eigenvalues for $(2,8)$ are $r = .05 \pm 0.61i$. Therefore, $(2,8)$ is an unstable spiral point.

(c)



21.(a) The critical points are solutions of

$$3\left(x+y-\frac{1}{3}x^3-k\right)=0$$
$$-\frac{1}{3}(x+0.8y-0.7)=0.$$

Setting $x'=0$ and solving for y yields $y=x^3/3-x+k$. Substituting this into $y'=0$ gives $W(x)=x+0.8(x^3/3-x+k)-0.7=0.8x^3/3+0.2x+(0.8k-0.7)=0$. Since $W'(x)=0.8x^2+0.2$ is never zero, we conclude that W always has a positive slope and thus the cubic equation crosses the x axis only once for all values of k. For that value of x, we can calculate y and will find one critical point.

(b) Using the equation in part (a) for x and setting $k=0$, we see that the x coordinate of the critical point is $x=1.19941$. Substituting that value for x into the second equation, we conclude that $y=-0.62426$. The Jacobian matrix is given by

$$\mathbf{J}=\begin{pmatrix}3-3x^2 & 3\\ -1/3 & -4/15\end{pmatrix}.$$

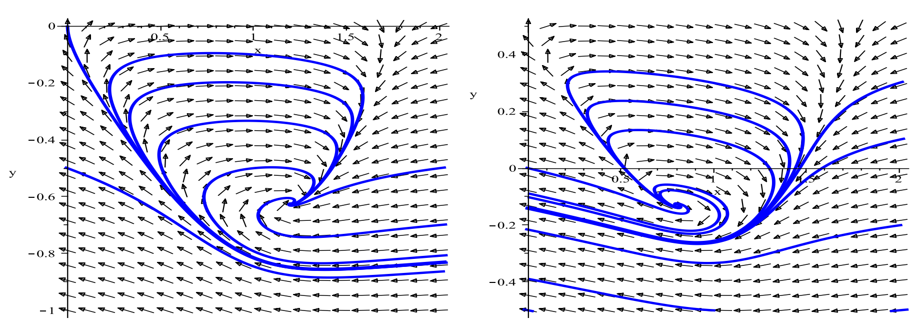
Therefore, at $(1.19941,-0.62426)$,

$$\mathbf{J}(1.19941,-0.62426)=\begin{pmatrix}-1.316 & 3\\ -1/3 & -4/15\end{pmatrix}.$$

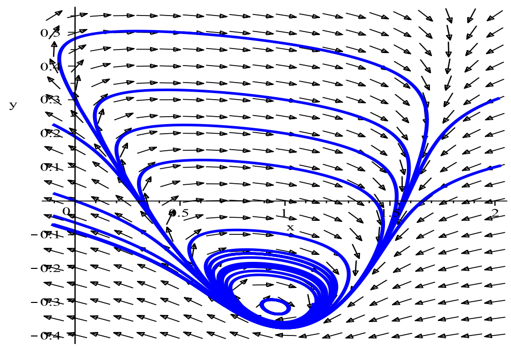
The eigenvalues of this matrix are $r=-0.791\pm 0.851i$. Therefore, the critical point $(1.19941,-0.62426)$ is an asymptotically stable spiral point.
Now using the equation in part (a) and setting $k=0.5$, we see that the $x-$coordinate of the critical point is $x=0.80485$. Substituting that value for x into the second equation, we conclude that $y=-0.13106$. Therefore, at $(0.80485,-0.13106)$,

$$\mathbf{J}(0.80485,-0.13106)=\begin{pmatrix}1.05665 & 3\\ -1/3 & -4/15\end{pmatrix}.$$

The eigenvalues of this matrix are $r=0.395\pm 0.7498i$. Therefore, the critical point $(0.80485,-0.13106)$ is an unstable spiral point.

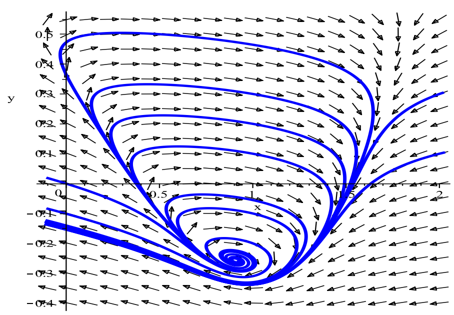


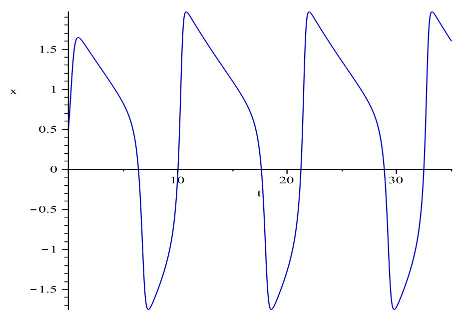
(c) Letting $k=0.1,\ 0.2,\ 0.3,\ 0.4$ in the cubic equation of part (a) and finding the corresponding eigenvalues from the matrix in part (b), we find that the real part of the eigenvalues changes sign between $k=0.3$ and $k=0.4$. Continuing in this fashion, we find that $k_0 \approx 0.3465$. For this value of k_0, we calculate the critical point as in part (b). In particular, we find that the critical point is $(0.9545, -0.31813)$.



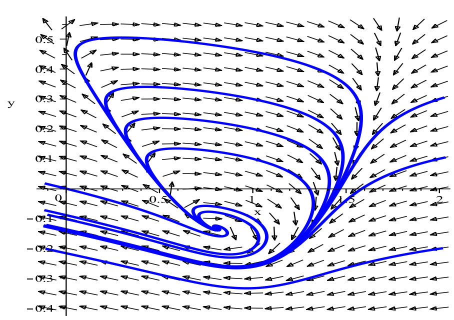
(d) In what follows, we consider initial conditions $x(0) = 0.5, y(0) = 0.5$.

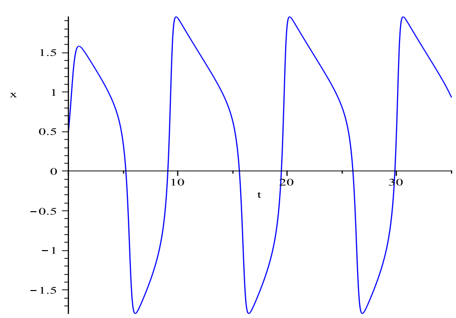
For $k = 0.4$, $T \approx 11.23$.



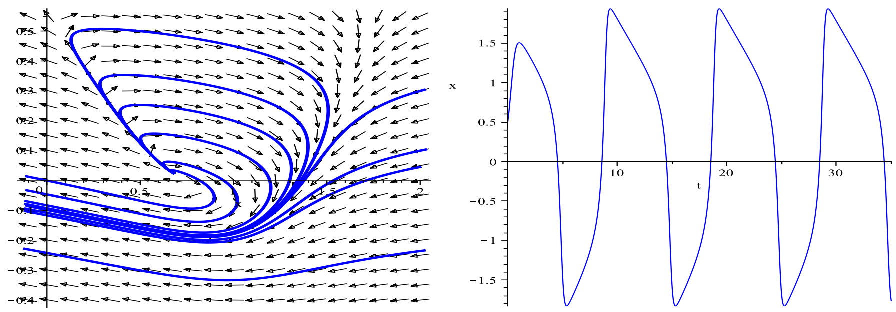


For $k = 0.5$, $T \approx 10.37$.





For $k = 0.6$, $T \approx 9.93$.



(e) Again, iterating as in part (c), we find for $k = 1.403$ that $x_0 = -0.9541$ and for $k = 1.404$ that $x_0 = -0.9549$. Substituting these values into the coefficient matrix of part (b) and finding the eigenvalues we find that the real part changes sign between $k = 1.403$ and $k = 1.404$. Thus the critical point again becomes asymptotically stable.

9.8

1.(a) Since the eigenvalues must be solutions of $-(8/3+\lambda)(\lambda^2+11\lambda-10(r-1)) = 0$, we see that the eigenvalues are given by $\lambda = -8/3$ and the roots of $\lambda^2+11\lambda-10(r-1) = 0$. The roots of the quadratic equation are $\lambda = (-11 \pm \sqrt{81+40r})/2$. Therefore, the eigenvalues are as stated in Eq.(10).

(b) The corresponding eigenvectors are given by $\boldsymbol{\xi}_1 = (0,0,1)^T$ for $\lambda_1 = -8/3$, $\boldsymbol{\xi}_2 = ((-9+\sqrt{81+40r})/2r, 1, 0)^T$ for $\lambda_2 = (-11+\sqrt{81+40r})/2$ and $\boldsymbol{\xi}_3 = ((-9-\sqrt{81+40r})/2r, 1, 0)^T$ for $\lambda_3 = (-11-\sqrt{81+40r})/2$

(c) For $r = 28$, using the formulas for the eigenvalues and eigenvectors from part (b), we see that $\lambda_1 = -8/3$ with eigenvector $\boldsymbol{\xi}_1 = (0,0,1)^T$, $\lambda_2 \approx 11.828$ with eigenvector $\boldsymbol{\xi}_2 \approx (20, 43.655, 0)^T$, and $\lambda_3 \approx -22.828$ with eigenvector $\boldsymbol{\xi}_3 \approx (20, -25.655, 0)^T$.

2.(a) Using the values $\sigma = 10$ and $b = 8/3$, the Jacobian matrix associated with our nonlinear system is

$$\mathbf{J} = \begin{pmatrix} -10 & 10 & 0 \\ r-z & -1 & -x \\ y & x & -8/3 \end{pmatrix}.$$

Therefore, at the critical point $P_2 = (\sqrt{8(r-1)/3}, \sqrt{8(r-1)/3}, r-1)$,

$$\mathbf{J}(P_2) = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{8(r-1)/3} \\ \sqrt{8(r-1)/3} & \sqrt{8(r-1)/3} & -8/3 \end{pmatrix},$$

as stated in Eq.(11).

(b) The eigenvalues of the system must satisfy $\det(A - \lambda I) = 0$ where $A = \mathbf{J}(P_2)$. We see that

$$A - \lambda I = \begin{pmatrix} -10-\lambda & 10 & 0 \\ 1 & -1-\lambda & -\sqrt{8(r-1)/3} \\ \sqrt{8(r-1)/3} & \sqrt{8(r-1)/3} & -8/3-\lambda \end{pmatrix}.$$

The determinant is

$$(-10-\lambda)[(1+\lambda)(8/3+\lambda) + (8(r-1)/3)] - 10[(-8/3-\lambda) + 8(r-1)/3] =$$

$$= (-10-\lambda)[\lambda^2 + 11\lambda/3 + 8/3 + 8(r-1)/3] + 10(8/3+\lambda) - 80(r-1)/3 =$$

$$= -10\lambda^2 - 110\lambda/3 - 80/3 - 80(r-1)/3 - \lambda^3 - 11\lambda^2/3 - 8\lambda/3 - 8(r-1)\lambda/3 +$$

$$+80/3 + 10\lambda - 80(r-1)/3 = -\lambda^3 - 41\lambda^2/3 - 8(r+10)\lambda/3 - 160(r-1)/3.$$

Setting this equation equal to zero and multiplying by -3, we arrive at Eq.(12).

(c) For $r = 28$, Eq.(12) becomes $3\lambda^3 + 41\lambda^2 + 304\lambda + 4320 = 0$. The solutions of this equation are $\lambda = -13.8546$, and $0.093956 \pm 10.1945i$.

3.(a) For $r = 28$, in Problem 2, we saw that the real part of the complex roots was positive. By numerical investigation, we see that the real part changes sign at $r \approx 24.737$.

(b) Suppose a cubic polynomial $x^3 + Ax^2 + Bx + C$ has one real zero and two pure imaginary zeros. Then the polynomial can be factored as $(x - \lambda_1)(x^2 + \lambda_2)$ where $\lambda_2 > 0$. Therefore,

$$x^3 + Ax^2 + Bx + C = (x - \lambda_1)(x^2 + \lambda_2) = x^3 - \lambda_1 x^2 + \lambda_2 x - \lambda_1\lambda_2,$$

which implies that $A = -\lambda_1$, $B = \lambda_2$, and $C = -\lambda_1\lambda_2 = AB$. Therefore, if $AB \neq C$, the cubic polynomial will not have the specified type of roots.

(c) First, we rewrite the equation as

$$\lambda^3 + \frac{41}{3}\lambda^2 + \frac{8(r+10)}{3}\lambda + \frac{160(r-1)}{3} = 0.$$

Using the result from part (b), we need to find when $AB = C$, where $A = 41/3$, $B = 8(r+10)/3$ and $C = 160(r-1)/3$. Setting $AB = C$, we have the equation $328(r+10)/9 = 160(r-1)/3$. Solving this equation, we see that the real part of the complex roots changes sign when $r = 470/19$.

4. We have

$$\dot{V} = 2x(\sigma(-x+y)) + 2\sigma y(rx - y - xz) + 2\sigma z(-bz + xy) =$$

$$-2\sigma x^2 + 2\sigma xy + 2\sigma rxy - 2\sigma y^2 - 2\sigma bz^2 = -2\sigma((x^2 - (r+1)xy + y^2) + bz^2).$$

For $r < 1$, the term inside the parentheses remains positive for all values of x and y (by Theorem 9.6.4), and thus $\dot{V}$ is negative definite. Thus, by the extension of Theorem 9.6.1 to three equations, we conclude that the origin is an asymptotically stable critical point.

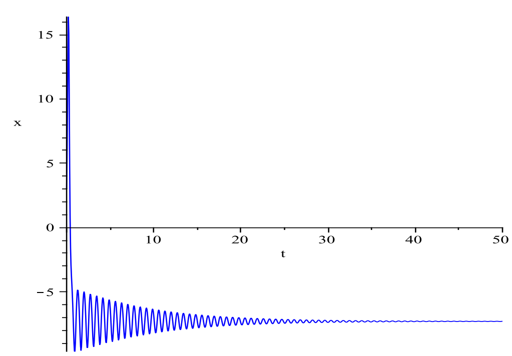
5.(a) We compute:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} = 2rx\frac{dx}{dt} + 2\sigma y\frac{dy}{dt} + 2\sigma(z-2r)\frac{dz}{dt} =$$
$$= 2\sigma rx(-x+y) + 2\sigma y(rx - y - xz) + 2\sigma(z-2r)(-bz+xy) =$$
$$= -2\sigma rx^2 - 2\sigma y^2 - 2\sigma bz^2 + 4\sigma rbz = -2\sigma[rx^2 + y^2 + bz^2 - 2rbz] =$$
$$= -2\sigma[rx^2 + y^2 + b(z-r)^2 - br^2].$$

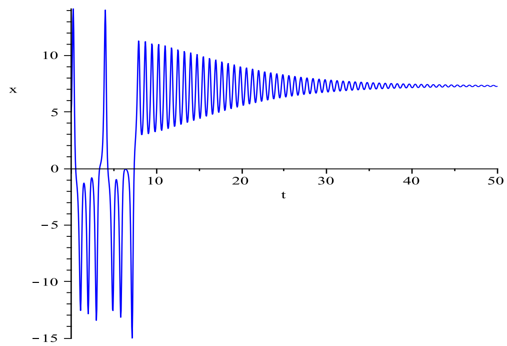
(b) From the proof of Theorem 9.6.1, we find that we need to show that $\dot{V}$, as found in part (a), is always negative as it crosses $V(x, y, z) = c$. (Actually, we need to use the extension of Theorem 9.6.1 to three equations, but the proof is very similar using the vector calculus approach.) From part (a) we see that $\dot{V} < 0$ whenever $rx^2 + y^2 + b(z-r)^2 > br^2$, which holds if (x, y, z) lies outside the ellipsoid $x^2/(br) + y^2/(br^2) + (z-r)^2/r^2 = 1$, Eq.(i). Thus we need to choose c such that $V = c$ lies outside Eq.(i). Writing $V = c$ in the form of Eq.(i) we obtain the ellipsoid $x^2/(c/r) + y^2/(c/\sigma) + (z-2r)^2/(c/\sigma) = 1$, Eq.(ii). Now let $M = \max(\sqrt{br}, r\sqrt{b}, r)$, then the ellipsoid (i) is contained inside the sphere $S1$: $x^2/M^2 + y^2/M^2 + (z-r)^2/M^2 = 1$. Let $S2$ be a sphere centered at $(0, 0, 2r)$ with radius $M + r$: $x^2/(M+r)^2 + y^2/(M+r)^2 + (z-2r)^2/(M+r)^2 = 1$, then $S1$ is contained in $S2$. Thus, if we choose c, in Eq.(ii), such that $c/r > (M+r)^2$ and $c/\sigma > (M+r)^2$, then $\dot{V} < 0$ as the trajectory crosses $V(x, y, z) = c$. Note that this is a sufficient condition and there may be many other better choices using different techniques.

(c) With the given values we obtain $c \approx 33,450.7$.

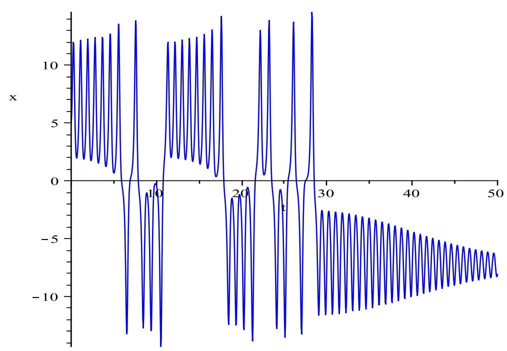
8.(a) $r = 21$, with initial point $(3, 8, 0)$:



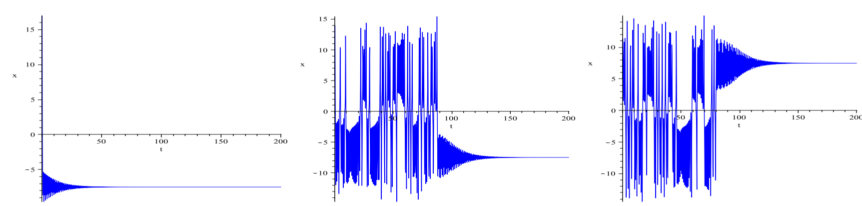
$r = 21$, with initial point $(5,\ 5\,,5)$:



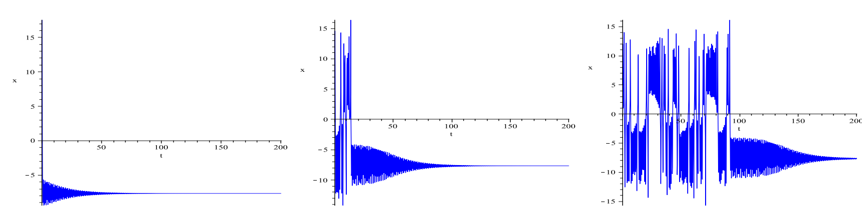
$r = 21$, with initial point $(5,\ 5\,,10)$:



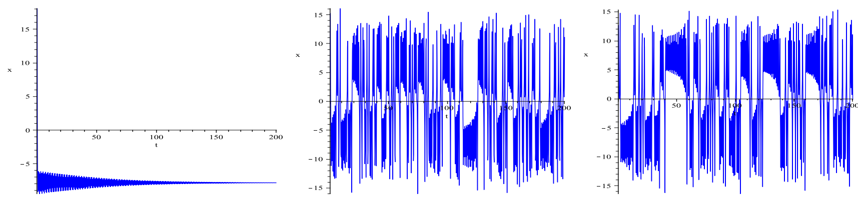
(b) $r = 22$, with initial points $(3\,,8\,,0)$, $(5,\ 5\,,5)$, $(5,\ 5\,,10)$:



$r = 23$, with initial points $(3\,,8\,,0)$, $(5,\ 5\,,5)$, $(5,\ 5\,,10)$:



$r = 24$, with initial points $(3\,,8\,,0)$, $(5,\,5\,,5)$, $(5,\,5\,,10)$:



11.(a) Let $a = 1/4$ and $b = 1/2$ in Eq.(i) to get $x' = -y - z$, $y' = x + y/4$ and $z' = 1/2 + z(x - c)$. Setting these equal to zero yields $x = z/4$ and $y = -z$ from the first two equations. Substitution into the third gives $z^2/4 - cz + 1/2 = 0$, and thus $z = 2(c \pm \sqrt{c^2 - 1/2})$, which gives the desired results.

(b) If we let $F = -y - z$, $G = x + ay$ and $H = b + z(x - c)$, then by Eq.(3) the Jacobian matrix is given by

$$\mathbf{J} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - c \end{pmatrix}.$$

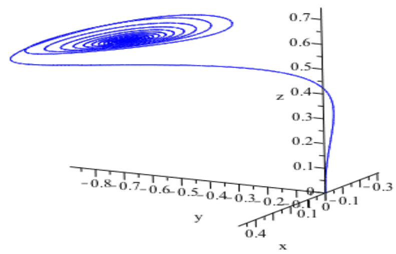
If we set $a = 1/4$, $b = 1/2$, $c = \sqrt{1/2}$, then $x = \sqrt{2}/4$, $y = -\sqrt{2}$, and $z = \sqrt{2}$. The eigenvalues for the Jacobian at this critical point are given by

$$\det(\mathbf{J} - r\mathbf{I}) = \begin{vmatrix} -r & -1 & -1 \\ 1 & 1/4 - r & 0 \\ \sqrt{2} & 0 & -\sqrt{2}/4 - r \end{vmatrix} =$$

$$-r(r^2 + r(\frac{1}{2\sqrt{2}} - \frac{1}{4}) + (1 + \frac{15}{8\sqrt{2}})) = 0.$$

One eigenvalue is 0, the other two can be found by solving the quadratic equation inside the parentheses above. We obtain a complex conjugate pair with negative real part. If $a = 1/4$, $b = 1/2$, $c = 1$, then we get two critical points: $x = (2 - \sqrt{2})/4$, $y = -2 + \sqrt{2}$, and $z = 2 - \sqrt{2}$ and $x = (2 + \sqrt{2})/4$, $y = -2 - \sqrt{2}$, and $z = 2 + \sqrt{2}$. The eigenvalues can be found again by finding the roots of $\det(\mathbf{J} - r\mathbf{I}) = 0$; in the first case we get one negative real and a complex conjugate pair with negative real part, in the second case we get a positive real and a complex conjugate pair with negative real part.

(c) With initial point $(0\,,0\,,0)$:



(d) With initial point $(1,\,1\,,1)$: