

4.4

2. The characteristic equation is $r(r^2 - 1) = 0$. Hence the homogeneous solution is $y_c(t) = c_1 + c_2e^t + c_3e^{-t}$. The Wronskian is evaluated as $W(1, e^t, e^{-t}) = 2$. Now compute the three determinants

$$W_1(t) = \begin{vmatrix} 0 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 1 & e^t & e^{-t} \end{vmatrix} = -2, \quad W_2(t) = \begin{vmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & 1 & e^{-t} \end{vmatrix} = e^{-t},$$

$$W_3(t) = \begin{vmatrix} 1 & e^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & 1 \end{vmatrix} = e^t.$$

The solution of the system of equations (10) is

$$u_1'(t) = \frac{tW_1(t)}{W(t)} = -t, \quad u_2'(t) = \frac{tW_2(t)}{W(t)} = te^{-t}/2,$$

$$u_3'(t) = \frac{tW_3(t)}{W(t)} = te^t/2.$$

Hence $u_1(t) = -t^2/2$, $u_2(t) = -e^{-t}(t+1)/2$, $u_3(t) = e^t(t-1)/2$. The particular solution becomes $Y(t) = -t^2/2 - (t+1)/2 + (t-1)/2 = -t^2/2 - 1$. The constant is a solution of the homogeneous equation, therefore the general solution is

$$y(t) = c_1 + c_2e^t + c_3e^{-t} - t^2/2.$$

3. From Problem 13 in Section 4.2, $y_c(t) = c_1e^{-t} + c_2e^t + c_3e^{2t}$. The Wronskian is evaluated as $W(e^{-t}, e^t, e^{2t}) = 6e^{2t}$. Now compute the three determinants

$$W_1(t) = \begin{vmatrix} 0 & e^t & e^{2t} \\ 0 & e^t & 2e^{2t} \\ 1 & e^t & 4e^{2t} \end{vmatrix} = e^{3t}, \quad W_2(t) = \begin{vmatrix} e^{-t} & 0 & e^{2t} \\ -e^{-t} & 0 & 2e^{2t} \\ e^{-t} & 1 & 4e^{2t} \end{vmatrix} = -3e^t,$$

$$W_3(t) = \begin{vmatrix} e^{-t} & e^t & 0 \\ -e^{-t} & e^t & 0 \\ e^{-t} & e^t & 1 \end{vmatrix} = 2.$$

Hence $u_1'(t) = e^{5t}/6$, $u_2'(t) = -e^{3t}/2$, $u_3'(t) = e^{2t}/3$. Therefore the particular solution can be expressed as $Y(t) = e^{-t} [e^{5t}/30] - e^t [e^{3t}/6] + e^{2t} [e^{2t}/6] = e^{4t}/30$.

6. From Problem 22 in Section 4.2, $y_c(t) = c_1 \cos t + c_2 \sin t + t [c_3 \cos t + c_4 \sin t]$. The Wronskian is evaluated as $W(\cos t, \sin t, t \cos t, t \sin t) = 4$. Now compute the four auxiliary determinants

$$W_1(t) = \begin{vmatrix} 0 & \sin t & t \cos t & t \sin t \\ 0 & \cos t & \cos t - t \sin t & \sin t + t \cos t \\ 0 & -\sin t & -2 \sin t - t \cos t & 2 \cos t - t \sin t \\ 1 & -\cos t & -3 \cos t + t \sin t & -3 \sin t - t \cos t \end{vmatrix} =$$

$$= -2 \sin t + 2t \cos t,$$

$$W_2(t) = \begin{vmatrix} \cos t & 0 & t \cos t & t \sin t \\ -\sin t & 0 & \cos t - t \sin t & \sin t + t \cos t \\ -\cos t & 0 & -2 \sin t - t \cos t & 2 \cos t - t \sin t \\ \sin t & 1 & -3 \cos t + t \sin t & -3 \sin t - t \cos t \end{vmatrix} =$$

$$= 2t \sin t + 2 \cos t,$$

$$W_3(t) = \begin{vmatrix} \cos t & \sin t & 0 & t \sin t \\ -\sin t & \cos t & 0 & \sin t + t \cos t \\ -\cos t & -\sin t & 0 & 2 \cos t - t \sin t \\ \sin t & -\cos t & 1 & -3 \sin t - t \cos t \end{vmatrix} = -2 \cos t,$$

$$W_4(t) = \begin{vmatrix} \cos t & \sin t & t \cos t & 0 \\ -\sin t & \cos t & \cos t - t \sin t & 0 \\ -\cos t & -\sin t & -2 \sin t - t \cos t & 0 \\ \sin t & -\cos t & -3 \cos t + t \sin t & 1 \end{vmatrix} = -2 \sin t.$$

It follows that

$$u_1'(t) = [-\sin^2 t + t \sin t \cos t] / 2, \quad u_2'(t) = [t \sin^2 t + \sin t, \cos t] / 2,$$

$$u_3'(t) = -\sin t \cos t / 2, \quad \text{and} \quad u_4'(t) = -\sin^2 t / 2.$$

Hence

$$u_1(t) = (3 \sin t \cos t - 2t \cos^2 t - t) / 8, \quad u_2(t) = (\sin^2 t - 2 \cos^2 t - 2t \sin t \cos t + t^2) / 8,$$

$$u_3(t) = -\sin^2 t / 4, \quad \text{and} \quad u_4(t) = [\cos t \sin t - t] / 4.$$

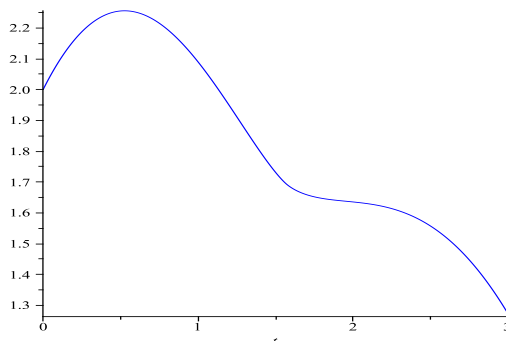
Therefore the particular solution can be expressed as $Y(t) = u_1(t) \cos t + u_2(t) \sin t + u_3(t)t \cos t + u_4(t)t \sin t = (\sin t - 3t \cos t - t^2 \sin t) / 8$. Note that only the last term is not a solution of the homogeneous equation. Hence the general solution is $y(t) = c_1 \cos t + c_2 \sin t + t [c_3 \cos t + c_4 \sin t] - t^2 \sin t / 8$.

8. Based on the results in Problem 2, $y_c(t) = c_1 + c_2 e^t + c_3 e^{-t}$. It was also shown that $W(1, e^t, e^{-t}) = 2$, with $W_1(t) = -2$, $W_2(t) = e^{-t}$, $W_3(t) = e^t$. Therefore we have $u_1'(t) = -\csc t$, $u_2'(t) = e^{-t} \csc t / 2$, $u_3'(t) = e^t \csc t / 2$. The particular solution can be expressed as $Y(t) = [u_1(t)] + e^{-t} [u_2(t)] + e^t [u_3(t)]$. More specifically,

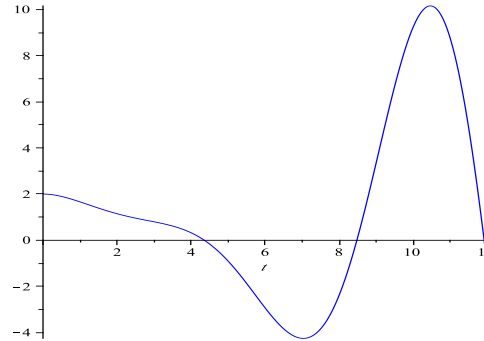
$$Y(t) = \ln |\csc(t) + \cot(t)| + \frac{e^t}{2} \int_{t_0}^t e^{-s} \csc(s) ds + \frac{e^{-t}}{2} \int_{t_0}^t e^s \csc(s) ds$$

$$= \ln |\csc(t) + \cot(t)| + \int_{t_0}^t \cosh(t-s) \csc(s) ds.$$

9. Based on Problem 4, $u_1'(t) = \sec t$, $u_2'(t) = -1$, $u_3'(t) = -\tan t$. The particular solution can be expressed as $Y(t) = [u_1(t)] + \cos t [u_2(t)] + \sin t [u_3(t)]$. That is, $Y(t) = \ln |\sec(t) + \tan(t)| - t \cos t + \sin t \ln |\cos(t)|$. Hence the general solution of the initial value problem is $y(t) = c_1 + c_2 \cos t + c_3 \sin t + \ln |\sec(t) + \tan(t)| - t \cos t + \sin t \ln |\cos(t)|$. Invoking the initial conditions, we require that $c_1 + c_2 = 2$, $c_3 = 1$, $-c_2 = -2$. Therefore $y(t) = 2 \cos t + \sin t + \ln |\sec(t) + \tan(t)| - t \cos t + \sin t \ln |\cos(t)|$. Since $-\pi/2 < t < \pi/2$, the absolute value signs may be removed.



10. From Problem 6, $y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t - t^2 \sin t / 8$. In order to satisfy the initial conditions, we require that $c_1 = 2$, $c_2 + c_3 = 0$, $-c_1 + 2c_4 = -1$, $-3/4 - c_2 - 3c_3 = 1$. Therefore $y(t) = (7 \sin t - 7t \cos t + 4t \sin t - t^2 \sin t) / 8 + 2 \cos t$.



12. From Problem 8, the general solution of the initial value problem is

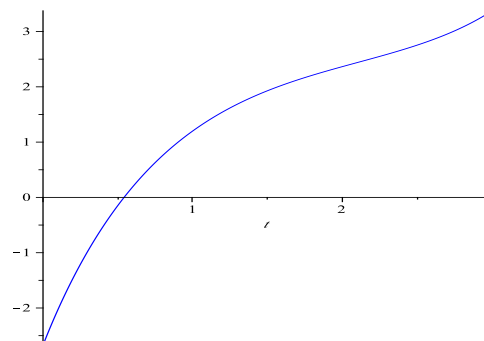
$$y(t) = c_1 + c_2 e^t + c_3 e^{-t} + \ln |\csc(t) + \cot(t)| + \frac{e^t}{2} \int_{t_0}^t e^{-s} \csc(s) ds + \frac{e^{-t}}{2} \int_{t_0}^t e^s \csc(s) ds.$$

In this case, $t_0 = \pi/2$. Observe that $y(\pi/2) = y_c(\pi/2)$, $y'(\pi/2) = y'_c(\pi/2)$, and $y''(\pi/2) = y''_c(\pi/2)$. Therefore we obtain the system of equations

$$\begin{aligned} c_1 + c_2 e^{\pi/2} + c_3 e^{-\pi/2} &= 2, \\ c_2 e^{\pi/2} - c_3 e^{-\pi/2} &= 1, \\ c_2 e^{\pi/2} + c_3 e^{-\pi/2} &= -1. \end{aligned}$$

Hence the solution of the initial value problem is

$$y(t) = 3 - e^{-t+\pi/2} + \ln |\csc(t) + \cot(t)| + \int_{\pi/2}^t \cosh(t-s) \csc(s) ds.$$



13. First write the equation as $y''' + x^{-1}y'' - 2x^{-2}y' + 2x^{-3}y = 2x$. The Wronskian is evaluated as $W(x, x^2, 1/x) = 6/x$. Now compute the three determinants

$$W_1(x) = \begin{vmatrix} 0 & x^2 & 1/x \\ 0 & 2x & -1/x^2 \\ 1 & 2 & 2/x^3 \end{vmatrix} = -3, \quad W_2(x) = \begin{vmatrix} x & 0 & 1/x \\ 1 & 0 & -1/x^2 \\ 0 & 1 & 2/x^3 \end{vmatrix} = 2/x,$$

$$W_3(x) = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2.$$

Hence $u_1'(x) = -x^2$, $u_2'(x) = 2x/3$, $u_3'(x) = x^4/3$. Therefore the particular solution can be expressed as

$$Y(x) = x[-x^3/3] + x^2[x^2/3] + \frac{1}{x}[x^5/15] = x^4/15.$$

15. The homogeneous solution is $y_c(t) = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t$. The Wronskian is evaluated as $W(\cos t, \sin t, \cosh t, \sinh t) = 4$. Now the four additional determinants are given by $W_1(t) = 2 \sin t$, $W_2(t) = -2 \cos t$, $W_3(t) = -2 \sinh t$, $W_4(t) = 2 \cosh t$. It follows that

$$\begin{aligned} u_1'(t) &= g(t) \sin(t)/2, & u_2'(t) &= -g(t) \cos(t)/2, \\ u_3'(t) &= -g(t) \sinh(t)/2, & u_4'(t) &= g(t) \cosh(t)/2. \end{aligned}$$

Therefore the particular solution can be expressed as

$$\begin{aligned} Y(t) &= \frac{\cos(t)}{2} \int_{t_0}^t g(s) \sin(s) ds - \frac{\sin(t)}{2} \int_{t_0}^t g(s) \cos(s) ds \\ &\quad - \frac{\cosh(t)}{2} \int_{t_0}^t g(s) \sinh(s) ds + \frac{\sinh(t)}{2} \int_{t_0}^t g(s) \cosh(s) ds. \end{aligned}$$

Using the appropriate identities, the integrals can be combined to obtain

$$Y(t) = \frac{1}{2} \int_{t_0}^t g(s) \sinh(t-s) ds - \frac{1}{2} \int_{t_0}^t g(s) \sin(t-s) ds.$$

17. First write the equation as $y''' - 3x^{-1}y'' + 6x^{-2}y' - 6x^{-3}y = g(x)/x^3$. It can be shown that $y_c(x) = c_1x + c_2x^2 + c_3x^3$ is a solution of the homogeneous equation. The Wronskian of this fundamental set of solutions is $W(x, x^2, x^3) = 2x^3$. The three additional determinants are given by $W_1(x) = x^4$, $W_2(x) = -2x^3$, $W_3(x) = x^2$. Hence $u_1'(x) = g(x)/2x^2$, $u_2'(x) = -g(x)/x^3$, $u_3'(x) = g(x)/2x^4$. Now the particular solution can be expressed as

$$\begin{aligned} Y(x) &= x \int_{x_0}^x \frac{g(t)}{2t^2} dt - x^2 \int_{x_0}^x \frac{g(t)}{t^3} dt + x^3 \int_{x_0}^x \frac{g(t)}{2t^4} dt = \\ &= \frac{1}{2} \int_{x_0}^x \left[\frac{x}{t^2} - \frac{2x^2}{t^3} + \frac{x^3}{t^4} \right] g(t) dt. \end{aligned}$$