4.2

1. The magnitude of $1+i$ is $R=\sqrt{2}$ and the polar angle is $\pi/4$. Hence the polar form is given by $1+i=\sqrt{2}\,e^{i\pi/4}$.

3. The magnitude of -3 is $R=3$ and the polar angle is π. Hence $-3=3\,e^{i\pi}$.

4. The magnitude of $-i$ is $R=1$ and the polar angle is $3\pi/2$. Hence $-i=\;e^{3\pi i/2}$.

5. The magnitude of $\sqrt{3}-i$ is $R=2$ and the polar angle is $-\pi/6=11\pi/6$. Hence the polar form is given by $\sqrt{3}-i=2\,e^{11\pi i/6}$.

6. The magnitude of $-1-i$ is $R=\sqrt{2}$ and the polar angle is $5\pi/4$. Hence the polar form is given by $-1-i=\sqrt{2}\,e^{5\pi i/4}$.

7. Writing the complex number in polar form, $1=e^{2m\pi i}$, where m may be any integer. Thus $1^{1/3}=e^{2m\pi i/3}$. Setting $m=0,1,2$ successively, we obtain the three roots as $1^{1/3}=1$, $1^{1/3}=e^{2\pi i/3}$, $1^{1/3}=e^{4\pi i/3}$. Equivalently, the roots can also be written as 1, $\cos(2\pi/3)+i\,\sin(2\pi/3)=(-1+i\sqrt{3})/2$, $\cos(4\pi/3)+i\sin(4\pi/3)=(-1-i\sqrt{3})/2$.

9. Writing the complex number in polar form, $1=e^{2m\pi i}$, where m may be any integer. Thus $1^{1/4}=e^{2m\pi i/4}$. Setting $m=0,1,2,3$ successively, we obtain the three roots as $1^{1/4}=1$, $1^{1/4}=e^{\pi i/2}$, $1^{1/4}=e^{\pi i}$, $1^{1/4}=e^{3\pi i/2}$. Equivalently, the roots can also be written as 1, $\cos(\pi/2)+i\,\sin(\pi/2)=i$, $\cos(\pi)+i\sin(\pi)=-1$, $\cos(3\pi/2)+i\sin(3\pi/2)=-i$.

10. In polar form, $2(\cos\,\pi/3+i\,\sin\,\pi/3)=2\,e^{i(\pi/3+2m\pi)}$, in which m is any integer. Thus $[2(\cos\,\pi/3+i\,\sin\,\pi/3)]^{1/2}=2^{1/2}\,e^{i(\pi/6+m\pi)}$. With $m=0$, one square root is given by $2^{1/2}\,e^{i\pi/6}=(\sqrt{3}+i)/\sqrt{2}$. With $m=1$, the other root is given by $2^{1/2}\,e^{i7\pi/6}=(-\sqrt{3}-i)/\sqrt{2}$.

11. The characteristic equation is $r^3-r^2-r+1=0$. The roots are $r=-1,1,1$. One root is repeated, hence the general solution is $y=c_1e^{-t}+c_2e^t+c_3te^t$.

13. The characteristic equation is $r^3-2r^2-r+2=0$, with roots $r=-1,1,2$. The roots are real and distinct, so the general solution is $y=c_1e^{-t}+c_2e^t+c_3e^{2t}$.

14. The characteristic equation can be written as $r^2(r^2-4r+4)=0$. The roots are $r=0,0,2,2$. There are two repeated roots, and hence the general solution is given by $y=c_1+c_2t+c_3e^{2t}+c_4te^{2t}$.

16. The characteristic equation can be written as $(r^2-1)(r^2-4)=0$. The roots are given by $r=\pm 1,\pm 2$. The roots are real and distinct, hence the general solution is $y=c_1e^{-t}+c_2e^t+c_3e^{-2t}+c_4e^{2t}$.

17. The characteristic equation can be written as $(r^2-1)^3=0$. The roots are given by $r=\pm 1$, each with multiplicity three. Hence the general solution is $y=c_1e^{-t}+c_2te^{-t}+c_3t^2e^{-t}+c_4e^t+c_5te^t+c_6t^2e^t$.

18. The characteristic equation can be written as $r^2(r^4-1)=0$. The roots are given by $r=0,0,\pm 1,\pm i$. The general solution is $y=c_1+c_2t+c_3e^{-t}+c_4e^t+c_5\cos\,t+c_6\sin\,t$.

19. The characteristic equation can be written as $r(r^4-3r^3+3r^2-3r+2)=0$. Examining the coefficients, it follows that $r^4-3r^3+3r^2-3r+2=(r-1)(r-2)(r^2+1)$. Hence the roots are $r=0,1,2,\pm i$. The general solution of the ODE is given by $y=c_1+c_2e^t+c_3e^{2t}+c_4\cos\,t+c_5\sin\,t$.

20. The characteristic equation can be written as $r(r^3-8)=0$, with roots $r=0$,

$2\,e^{2m\pi i/3}$, $m = 0, 1, 2$. That is, $r = 0, 2, -1 \pm i\sqrt{3}$. Hence the general solution is $y = c_1 + c_2e^{2t} + e^{-t}\left[c_3 \cos\sqrt{3}\,t + c_4 \sin\sqrt{3}\,t\right]$.

21. The characteristic equation can be written as $(r^4 + 4)^2 = 0$. The roots of the equation $r^4 + 4 = 0$ are $r = 1 \pm i$, $-1 \pm i$. Each of these roots has multiplicity two. The general solution is $y = e^t\,[c_1 \cos t + c_2 \sin t\,] + te^t\,[c_3 \cos t + c_4 \sin t\,] + e^{-t}\,[c_5 \cos t + c_6 \sin t\,] + te^{-t}\,[c_7 \cos t + c_8 \sin t\,]$.

22. The characteristic equation can be written as $(r^2 + 1)^2 = 0$. The roots are given by $r = \pm i$, each with multiplicity two. The general solution is $y = c_1 \cos t + c_2 \sin t + t\,[c_3 \cos t + c_4 \sin t\,]$.

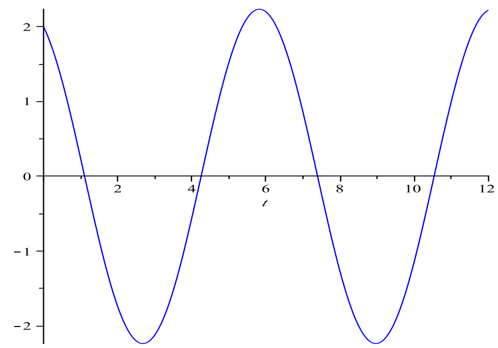
24. The characteristic equation is $r^3 + 5r^2 + 6r + 2 = 0$. Examining the coefficients, we find that $r^3 + 5r^2 + 6r + 2 = (r + 1)(r^2 + 4r + 2)$. Hence the roots are deduced as $r = -1$, $-2 \pm \sqrt{2}$. The general solution is $y = c_1e^{-t} + c_2e^{(-2+\sqrt{2})t} + c_3e^{(-2-\sqrt{2})t}$.

25. The characteristic equation is $18r^3 + 21r^2 + 14r + 4 = 0$. By examining the first and last coefficients, we find that $18r^3 + 21r^2 + 14r + 4 = (2r + 1)(9r^2 + 6r + 4)$. Hence the roots are $r = -1/2$, $(-1 \pm \sqrt{3}\,i)/3$. The general solution of the ODE is given by $y = c_1e^{-t/2} + e^{-t/3}\left[c_2 \cos(t/\sqrt{3}\,) + c_3 \sin(t/\sqrt{3}\,)\right]$.

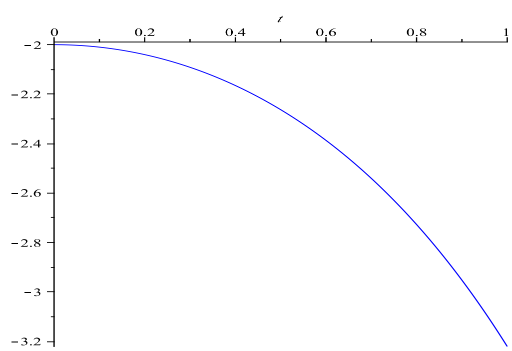
26. The characteristic equation is $r^4 - 7r^3 + 6r^2 + 30r - 36 = 0$. By examining the first and last coefficients, we find that $r^4 - 7r^3 + 6r^2 + 30r - 36 = (r - 3)(r + 2)(r^2 - 6r + 6)$. The roots are $r = -2, 3, 3 \pm \sqrt{3}$. The general solution is $y = c_1e^{-2t} + c_2e^{3t} + c_3e^{(3-\sqrt{3})t} + c_4e^{(3+\sqrt{3})t}$.

28. The characteristic equation is $r^4 + 6r^3 + 17r^2 + 22r + 14 = 0$. It can be shown that $r^4 + 6r^3 + 17r^2 + 22r + 14 = (r^2 + 2r + 2)(r^2 + 4r + 7)$. Hence the roots are $r = -1 \pm i$, $-2 \pm i\sqrt{3}$. The general solution of the euqation is $y = e^{-t}(c_1 \cos t + c_2 \sin t) + e^{-2t}(c_3 \cos\sqrt{3}\,t + c_4 \sin\sqrt{3}\,t)$.

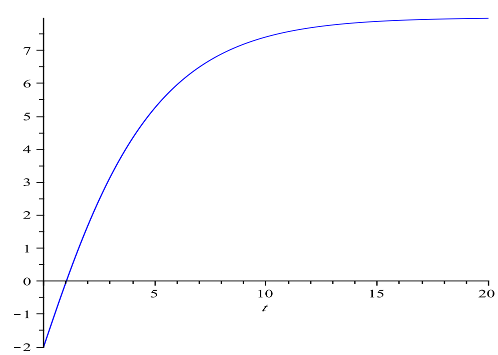
32. The characteristic equation is $r^3 - r^2 + r - 1 = 0$, with roots $r = 1$, $\pm i$. Hence the general solution is $y(t) = c_1e^t + c_2 \cos t + c_3 \sin t$. Invoking the initial conditions, we obtain the system of equations $c_1 + c_2 = 2$, $c_1 + c_3 = -1$, $c_1 - c_2 = -2$, with solution $c_1 = 0$, $c_2 = 2$, $c_3 = -1$. Therefore the solution of the initial value problem is $y(t) = 2 \cos t - \sin t$, which oscillates as $t \to \infty$.



33. The characteristic equation is $2r^4 - r^3 - 9r^2 + 4r + 4 = 0$, with roots $r = -1/2\,, 1\,, \pm 2$. Hence the general solution is $y(t) = c_1e^{-t/2} + c_2e^t + c_3e^{-2t} + c_4e^{2t}$. Applying the initial conditions, we obtain the system of equations $c_1 + c_2 + c_3 + c_4 = -2$, $-c_1/2 + c_2 - 2c_3 + 2c_4 = 0$, $c_1/4 + c_2 + 4c_3 + 4c_4 = -2$, $-c_1/8 + c_2 - 8c_3 + 8c_4 = 0$, with solution $c_1 = -16/15\,, c_2 = -2/3\,, c_3 = -1/6\,, c_4 = -1/10$. Therefore the solution of the initial value problem is $y(t) = -(16/15)e^{-t/2} - (2/3)e^t - e^{-2t}/6 - e^{2t}/10$. The solution decreases without bound.



35. The characteristic equation is $6\,r^3 + 5r^2 + r = 0$, with roots $r = 0\,, -1/3\,, -1/2$. The general solution is $y(t) = c_1 + c_2e^{-t/3} + c_3e^{-t/2}$. Invoking the initial conditions, we require that $c_1 + c_2 + c_3 = -2$, $-c_2/3 - c_3/2 = 2$, $c_2/9 + c_3/4 = 0$. The solution is $c_1 = 8\,, c_2 = -18\,, c_3 = 8$. Therefore the solution of the initial value problem is $y(t) = 8 - 18e^{-t/3} + 8e^{-t/2}$. It approaches 8 as $t \to \infty$.



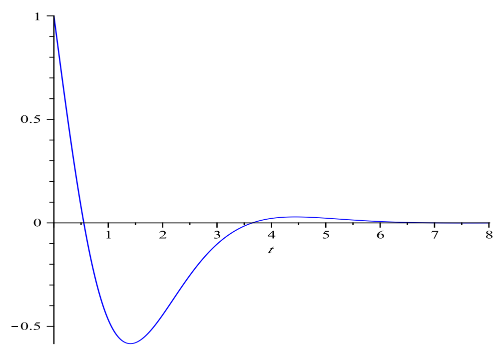
36. The general solution is derived in Problem 28 as

$$y(t) = e^{-t}\left[c_1 \cos t + c_2 \sin t\right] + e^{-2t}\left[c_3 \cos \sqrt{3}\,t + c_4 \sin \sqrt{3}\,t\right].$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_3 &= 1 \\ -c_1 + c_2 - 2c_3 + \sqrt{3}\,c_4 &= -2 \\ -2c_2 + c_3 - 4\sqrt{3}\,c_4 &= 0 \\ 2c_1 + 2c_2 + 10c_3 + 9\sqrt{3}\,c_4 &= 3 \end{aligned}$$

with solution $c_1 = 21/13$, $c_2 = -38/13$, $c_3 = -8/13$, $c_4 = 17\sqrt{3}/39$.



The solution is a rapidly decaying oscillation.

40.(a) Suppose that $c_1 e^{r_1 t} + c_2 e^{r_2 t} + \ldots + c_n e^{r_n t} = 0$, and each of the r_k are real and different. Multiplying this equation by $e^{-r_1 t}$, we obtain that $c_1 + c_2 e^{(r_2 - r_1)t} + \ldots + c_n e^{(r_n - r_1)t} = 0$. Differentiation results in

$$c_2(r_2 - r_1)e^{(r_2 - r_1)t} + \ldots + c_n(r_n - r_1)e^{(r_n - r_1)t} = 0\,.$$

(b) Now multiplying the latter equation by $e^{-(r_2 - r_1)t}$, and differentiating, we obtain

$$c_3(r_3 - r_2)(r_3 - r_1)e^{(r_3 - r_2)t} + \ldots + c_n(r_n - r_2)(r_n - r_1)e^{(r_n - r_2)t} = 0\,.$$

(c) Following the above steps in a similar manner, it follows that

$$c_n(r_n - r_{n-1})\ldots(r_n - r_1)e^{(r_n - r_{n-1})t} = 0\,.$$

Since these equations hold for all t, and all the r_k are different, we have $c_n = 0$. Hence $c_1 e^{r_1 t} + c_2 e^{r_2 t} + \ldots + c_{n-1} e^{r_{n-1} t} = 0\,, \quad -\infty < t < \infty$.

(d) The same procedure can now be repeated, successively, to show that $c_1 = c_2 = \ldots = c_n = 0$.

41.(a) Recall the derivative formula

$$\frac{d^n}{dx^n}(uv) = \binom{n}{0} v \frac{d^n u}{dx^n} + \binom{n}{1} \frac{dv}{dx} \frac{d^{n-1}u}{dx^{n-1}} + \ldots + \binom{n}{n} \frac{d^n v}{dx^n}\, u \;.$$

Let $u = (r - r_1)^s$ and $v = q(r)$. Note that

$$\frac{d^n}{dr^n}\left[(r-r_1)^s\right] = s \cdot (s-1) \ldots (s-n+1)(r-r_1)^{s-n}$$

and

$$\frac{d^s}{dr^s}\left[(r-r_1)^s\right] = s\,! \;.$$

Therefore

$$\frac{d^n}{dr^n}\left[(r-r_1)^s q(r)\right]\Big|_{r=r_1} = 0$$

only if $n < s$, since it is assumed that $q(r_1) \neq 0$.

(b) Differential operators commute, so that

$$\frac{\partial}{\partial r}\left(\frac{d^k}{dt^k}\, e^{rt}\right) = \frac{d^k}{dt^k}\left(\frac{\partial\, e^{rt}}{\partial r}\right) = \frac{d^k}{dt^k}(t\, e^{rt}).$$

Likewise,

$$\frac{\partial^j}{\partial r^j}\left(\frac{d^k}{dt^k}\, e^{rt}\right) = \frac{d^k}{dt^k}\left(\frac{\partial^j\, e^{rt}}{\partial r^j}\right) = \frac{d^k}{dt^k}(t^j\; e^{rt}).$$

It follows that

$$\frac{\partial^j}{\partial r^j} L\left[e^{rt}\right] = L\left[t^j\; e^{rt}\right].$$

(c) From Eq. (i), we have

$$\frac{\partial^j}{\partial r^j}\left[e^{rt}\; \mathrm{Z}(r)\right] = L\left[t^j\; e^{rt}\right].$$

Based on the product formula in part (a),

$$\frac{\partial^j}{\partial r^j}\left[e^{rt}\; \mathrm{Z}(r)\right]\Big|_{r=r_1} = 0$$

if $j < s$. Therefore $L\left[t^j\; e^{r_1 t}\right] = 0$ if $j < s$.

[redacted]