CHAPTER

4

Higher Order Linear Equations

4.1

1. The differential equation is in standard form. Its coefficients, as well as the function $g(t) = t$, are continuous everywhere. Hence solutions are valid on the entire real line.

3. Writing the equation in standard form, the coefficients are rational functions with singularities at $t = 0$ and $t = 1$. Hence the solutions are valid on the intervals $(-\infty,0)$, $(0\,,1)$, and $(1\,,\infty)$.

5. Writing the equation in standard form, the coefficients are rational functions with a singularity at $x_0 = 1$. Furthermore, $p_4(x) = \tan x/(x-1)$ is undefined, and hence not continuous, at $x_k = \pm(2k+1)\pi/2$, $k = 0, 1, 2, \ldots$. Hence solutions are defined on any interval that does not contain x_0 or x_k.

6. Writing the equation in standard form, the coefficients are rational functions with singularities at $x = \pm\, 2$. Hence the solutions are valid on the intervals $(-\infty, -\,2)$, $(-2\,, 2)$, and $(2\,,\infty)$.

7. Evaluating the Wronskian of the three functions, $W(f_1, f_2, f_3) = -14$. Hence the functions are linearly independent.

9. Evaluating the Wronskian of the four functions, $W(f_1, f_2, f_3, f_4) = 0$. Hence the functions are linearly dependent. To find a linear relation among the functions,

we need to find constants c_1, c_2, c_3, c_4, not all zero, such that

$$c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t) + c_4 f_4(t) = 0\,.$$

Collecting the common terms, we obtain

$$(c_2 + 2c_3 + c_4)t^2 + (2c_1 - c_3 + c_4)t + (-3c_1 + c_2 + c_4) = 0\,,$$

which results in three equations in four unknowns. Arbitrarily setting $c_4 = -1$, we can solve the equations $c_2 + 2c_3 = 1$, $2c_1 - c_3 = 1$, $-3c_1 + c_2 = 1$, to find that $c_1 = 2/7$, $c_2 = 13/7$, $c_3 = -3/7$. Hence

$$2f_1(t) + 13f_2(t) - 3f_3(t) - 7f_4(t) = 0\,.$$

10. Evaluating the Wronskian of the three functions, $W(f_1, f_2, f_3) = 156$. Hence the functions are linearly independent.

11. Substitution verifies that the functions are solutions of the differential equation. Furthermore, we have $W(1, \cos t, \sin t) = 1$.

12. Substitution verifies that the functions are solutions of the differential equation. Furthermore, we have $W(1, t, \cos t, \sin t) = 1$.

14. Substitution verifies that the functions are solutions of the differential equation. Furthermore, we have $W(1, t, e^{-t}, t\,e^{-t}) = e^{-2t}$.

15. Substitution verifies that the functions are solutions of the differential equation. Furthermore, we have $W(1, x, x^3) = 6x$.

16. Substitution verifies that the functions are solutions of the differential equation. Furthermore, we have $W(x, x^2, 1/x) = 6/x$.

18. The operation of taking a derivative is linear, and hence $(c_1 y_1 + c_2 y_2)^{(k)} = c_1 y_1^{(k)} + c_2 y_2^{(k)}$. It follows that

$$L\,[c_1 y_1 + c_2 y_2] = c_1 y_1^{(n)} + c_2 y_2^{(n)} + p_1(c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)}) + \ldots + p_n(c_1 y_1 + c_2 y_2).$$

Rearranging the terms, we obtain $L\,[c_1 y_1 + c_2 y_2] = c_1 L\,[y_1] + c_2 L\,[y_2]$. Since y_1 and y_2 are solutions, $L\,[c_1 y_1 + c_2 y_2] = 0$. The rest follows by induction.

20.(a) Let $f(t)$ and $g(t)$ be arbitrary functions. Then $W(f, g) = fg' - f'g$. Hence $W'(f, g) = f'g' + fg'' - f''g - f'g' = fg'' - f''g$. That is,

$$W'(f, g) = \begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix}.$$

Now expand the 3-by-3 determinant as

$$W(y_1, y_2, y_3) = y_1 \begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2 \begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3 \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}.$$

Differentiating, we obtain

$$W'(y_1, y_2, y_3) = y_1' \begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2' \begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3' \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} +$$
$$+ y_1 \begin{vmatrix} y_2' & y_3' \\ y_2''' & y_3''' \end{vmatrix} - y_2 \begin{vmatrix} y_1' & y_3' \\ y_1''' & y_3''' \end{vmatrix} + y_3 \begin{vmatrix} y_1' & y_2' \\ y_1''' & y_2''' \end{vmatrix}.$$

The second line follows from the observation above. Now we find that

$$W'(y_1, y_2, y_3) = \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hence the assertion is true, since the first determinant is equal to zero.

(b) Based on the properties of determinants,

$$p_2(t)p_3(t)W' = \begin{vmatrix} p_3\, y_1 & p_3\, y_2 & p_3\, y_3 \\ p_2\, y_1' & p_2\, y_2' & p_2\, y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Adding the first two rows to the third row does not change the value of the determinant. Since the functions are assumed to be solutions of the given ODE, addition of the rows results in

$$p_2(t)p_3(t)W' = \begin{vmatrix} p_3\, y_1 & p_3\, y_2 & p_3\, y_3 \\ p_2\, y_1' & p_2\, y_2' & p_2\, y_3' \\ -p_1\, y_1'' & -p_1\, y_2'' & -p_1\, y_3'' \end{vmatrix}.$$

It follows that $p_2(t)p_3(t)W' = -p_1(t)p_2(t)p_3(t)W$. As long as the coefficients are not zero, we obtain $W' = -p_1(t)W$.

(c) The first order equation $W' = -p_1(t)W$ is linear, with integrating factor $\mu(t) = e^{\int p_1(t)dt}$. Hence $W(t) = c\, e^{-\int p_1(t)dt}$. Furthermore, $W(t)$ is zero only if $c = 0$.

(d) It can be shown, by mathematical induction, that

$$W'(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \\ y_1' & y_2' & \cdots & y_{n-1}' & y_n' \\ \vdots & & & & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_{n-1}^{(n)} & y_n^{(n)} \end{vmatrix}.$$

Based on the reasoning in part (b), it follows that

$$p_2(t)p_3(t)\ldots p_n(t)W' = -p_1(t)p_2(t)p_3(t)\ldots p_n(t)W,$$

and hence $W' = -p_1(t)W$.

21. Inspection of the coefficients reveals that $p_1(t) = 2$. Based on Problem 20, we find that $W' = -2W$, and hence $W = ce^{-2t}$.

22. Inspection of the coefficients reveals that $p_1(t) = 0$. Based on Problem 20, we find that $W' = 0$, and hence $W = c$.

24. Writing the equation in standard form, we find that $p_1(t) = 1/t$. Using Abel's formula, the Wronskian has the form $W(t) = c\,e^{-\int 1/t\,dt} = ce^{-\ln t} = c/t$.

26. Let $y(t) = y_1(t)v(t)$. Then $y' = y_1'v + y_1v'$, $y'' = y_1''v + 2y_1'v' + y_1v''$, and $y''' = y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v'''$. Substitution into the ODE results in

$$y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v''' + p_1\left[y_1''v + 2y_1'v' + y_1v''\right] +$$
$$+p_2\left[y_1'v + y_1v'\right] + p_3y_1v = 0.$$

Since y_1 is assumed to be a solution, all terms containing the factor $v(t)$ vanish. Hence

$$y_1v''' + \left[p_1y_1 + 3y_1'\right]v'' + \left[3y_1'' + 2p_1y_1' + p_2y_1\right]v' = 0\,,$$

which is a second order ODE in the variable $u = v'$.

28. First write the equation in standard form:

$$y''' - 3\frac{t+2}{t(t+3)}y'' + 6\frac{t+1}{t^2(t+3)}y' - \frac{6}{t^2(t+3)}y = 0\,.$$

Let $y(t) = t^2v(t)$. Substitution into the given ODE results in

$$t^2v''' + 3\frac{t(t+4)}{t+3}v'' = 0\,.$$

Set $w = v''$. Then w is a solution of the first order differential equation

$$w' + 3\frac{t+4}{t(t+3)}w = 0\,.$$

This equation is linear, with integrating factor $\mu(t) = t^4/(t+3)$. The general solution is $w = c(t+3)/t^4$. Integrating twice, $v(t) = c_1t^{-1} + c_1t^{-2} + c_2t + c_3$. Hence $y(t) = c_1t + c_1 + c_2t^3 + c_3t^2$. Finally, since $y_1(t) = t^2$ and $y_2(t) = t^3$ are given solutions, the third independent solution is $y_3(t) = c_1t + c_1$.

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