3.4

2. The characteristic equation is $9r^2 + 6r + 1 = 0$, with the double root $r = -1/3$. The general solution is $y(t) = c_1 e^{-t/3} + c_2 t\, e^{-t/3}$.

3. The characteristic equation is $4r^2 - 4r - 3 = 0$, with roots $r = -1/2\,,\,3/2$. The general solution is $y(t) = c_1 e^{-t/2} + c_2 e^{3t/2}$.

4. The characteristic equation is $4r^2 + 12r + 9 = 0$, with double root $r = -3/2$. The general solution is $y(t) = (c_1 + c_2\, t)e^{-3t/2}$.

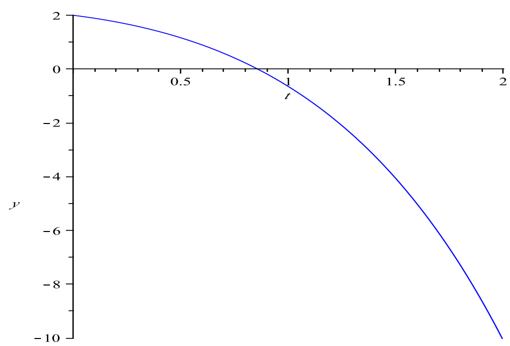
6. The characteristic equation is $r^2 - 6r + 9 = 0$, with the double root $r = 3$. The general solution is $y(t) = c_1 e^{3t} + c_2 t\, e^{3t}$.

7. The characteristic equation is $4r^2 + 17r + 4 = 0$, with roots $r = -1/4\,,\ -4$. The general solution is $y(t) = c_1 e^{-t/4} + c_2 e^{-4t}$.

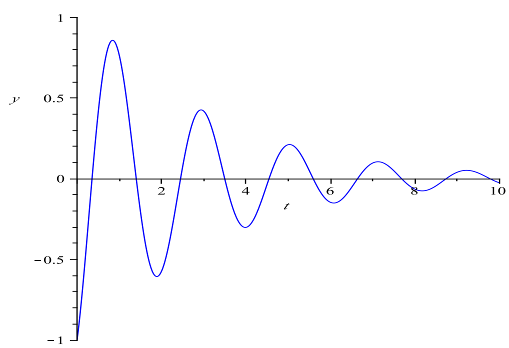
8. The characteristic equation is $16r^2 + 24r + 9 = 0$, with double root $r = -3/4$. The general solution is $y(t) = c_1 e^{-3t/4} + c_2 t\, e^{-3t/4}$.

10. The characteristic equation is $2r^2 + 2r + 1 = 0$. We obtain the complex roots $r = (-1 \pm i)/2$. The general solution is $y(t) = c_1 e^{-t/2}\cos(t/2) + c_2 e^{-t/2}\sin(t/2)$.

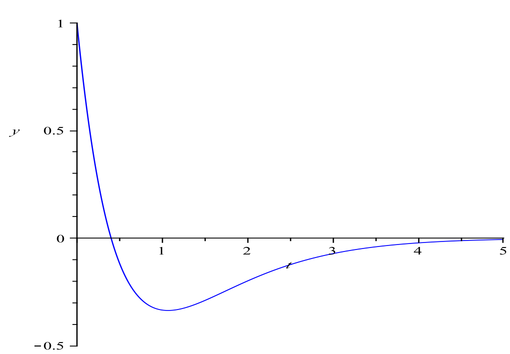
11. The characteristic equation is $9r^2 - 12r + 4 = 0$, with the double root $r = 2/3$. The general solution is $y(t) = c_1 e^{2t/3} + c_2 t\, e^{2t/3}$. Invoking the first initial condition, it follows that $c_1 = 2$. Now $y'(t) = (4/3 + c_2)e^{2t/3} + 2c_2 t\, e^{2t/3}/3$. Invoking the second initial condition, $4/3 + c_2 = -1$, or $c_2 = -7/3$. Hence we obtain the solution $y(t) = 2e^{2t/3} - (7/3)te^{2t/3}$. Since the second term dominates for large t, $y(t) \to -\infty$.



13. The characteristic equation is $9r^2 + 6r + 82 = 0$. We obtain the complex roots $r = -1/3 \pm 3\,i$. The general solution is $y(t) = c_1 e^{-t/3}\cos 3t + c_2 e^{-t/3}\sin 3t$. Based on the first initial condition, $c_1 = -1$. Invoking the second initial condition, we conclude that $1/3 + 3c_2 = 2$, or $c_2 = 5/9$. Hence $y(t) = -e^{-t/3}\cos 3t + (5/9)e^{-t/3}\sin 3t$. The solution oscillates with an exponentially decreasing amplitude.



15.(a) The characteristic equation is $4r^2 + 12r + 9 = 0$, with double root $r = -3/2$. The general solution is $y(t) = c_1 e^{-3t/2} + c_2 t\, e^{-3t/2}$. Invoking the first initial condition, it follows that $c_1 = 1$. Now $y'(t) = (-3/2 + c_2)e^{-3t/2} - (3/2)c_2 t\, e^{-3t/2}$. The second initial condition requires that $-3/2 + c_2 = -4$, or $c_2 = -5/2$. Hence the specific solution is $y(t) = e^{-3t/2} - (5/2)t\, e^{-3t/2}$.



(b) The solution crosses the x-axis at $t = 2/5$.

(c) The solution has a minimum at the point $(16/15\,, -5e^{-8/5}/3)$.

(d) Given that $y'(0) = b$, we have $-3/2 + c_2 = b$, or $c_2 = b + 3/2$. Hence the solution is $y(t) = e^{-3t/2} + (b + 3/2)t\,e^{-3t/2}$. Since the second term dominates, the long-term solution depends on the sign of the coefficient $b + 3/2$. The critical value is $b = -3/2$.

16. The characteristic roots are $r_1 = r_2 = 1/2$. Hence the general solution is given by $y(t) = c_1 e^{t/2} + c_2 t\,e^{t/2}$. Invoking the initial conditions, we require that $c_1 = 2$, and that $1 + c_2 = b$. The specific solution is $y(t) = 2e^{t/2} + (b-1)t\,e^{t/2}$. Since the second term dominates, the long-term solution depends on the sign of the coefficient $b - 1$. The critical value is $b = 1$.

18.(a) The characteristic roots are $r_1 = r_2 = -2/3$. Therefore the general solution is given by $y(t) = c_1 e^{-2t/3} + c_2 t\,e^{-2t/3}$. Invoking the initial conditions, we require that $c_1 = a$, and that $-2a/3 + c_2 = -1$. After solving for the coefficients, the specific solution is $y(t) = ae^{-2t/3} + (2a/3 - 1)t\,e^{-2t/3}$.

(b) Since the second term dominates, the long-term solution depends on the sign of the coefficient $2a/3 - 1$. The critical value is $a = 3/2$.

20.(a) The characteristic equation is $r^2 + 2a\,r + a^2 = (r + a)^2 = 0$.

(b) With $p(t) = 2a$, Abel's Formula becomes $W(y_1\,, y_2) = c\,e^{-\int 2a\,dt} = c\,e^{-2at}$.

(c) $y_1(t) = e^{-at}$ is a solution. From part (b), with $c = 1$, $e^{-at}\,y_2'(t) + a\,e^{-at}y_2(t) = e^{-2at}$, which can be written as $(e^{at}\,y_2(t))' = 1$, resulting in $e^{at}\,y_2(t) = t$.

22.(a) If the characteristic equation $ar^2 + br + c$ has equal roots r_1, then $ar_1^2 + br_1 + c = a(r - r_1)^2 = 0$. Then clearly $L[e^{rt}] = (ar^2 + br + c)e^{rt} = a(r - r_1)^2 e^{rt}$. This gives immediately that $L[e^{r_1 t}] = 0$.

(b) Differentiating the identity in part (a) with respect to r we get $(2ar + b)e^{rt} + (ar^2 + br + c)te^{rt} = 2a(r - r_1)e^{rt} + a(r - r_1)^2 te^{rt}$. Again, this gives $L[te^{r_1 t}] = 0$.

23. Set $y_2(t) = t^2\,v(t)$. Substitution into the differential equation results in

$$t^2(t^2 v'' + 4tv' + 2v) - 4t(t^2 v' + 2tv) + 6t^2 v = 0\,.$$

After collecting terms, we end up with $t^4 v'' = 0$. Hence $v(t) = c_1 + c_2 t$, and thus $y_2(t) = c_1 t^2 + c_2 t^3$. Setting $c_1 = 0$ and $c_2 = 1$, we obtain $y_2(t) = t^3$.

24. Set $y_2(t) = t\,v(t)$. Substitution into the differential equation results in

$$t^2(tv'' + 2v') + 2t(tv' + v) - 2tv = 0\,.$$

After collecting terms, we end up with $t^3 v'' + 4t^2 v' = 0$. This equation is linear in the variable $w = v'$. It follows that $v'(t) = c\,t^{-4}$, and $v(t) = c_1 t^{-3} + c_2$. Thus

$y_2(t) = c_1 t^{-2} + c_2 t$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(t) = t^{-2}$.

26. Set $y_2(t) = tv(t)$. Substitution into the differential equation results in $v'' - v' = 0$. This equation is linear in the variable $w = v'$. It follows that $v'(t) = c_1 e^t$, and $v(t) = c_1 e^t + c_2$. Thus $y_2(t) = c_1 te^t + c_2 t$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(t) = te^t$.

28. Set $y_2(x) = e^x v(x)$. Substitution into the differential equation results in $v'' + (x-2)/(x-1)\, v' = 0$. This equation is linear in the variable $w = v'$. An integrating factor is $\mu = e^{\int (x-2)/(x-1)\, dx} = e^x/(x-1)$. Rewrite the equation as $[e^x v'/(x-1)]' = 0$, from which it follows that $v'(x) = c(x-1)e^{-x}$. Hence $v(x) = c_1 x e^{-x} + c_2$ and $y_2(x) = c_1 x + c_2 e^x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x$.

29. Set $y_2(x) = y_1(x)\, v(x)$, in which $y_1(x) = x^{1/4} e^{2\sqrt{x}}$. It can be verified that y_1 is a solution of the differential equation, that is, $x^2 y_1'' - (x - 0.1875)y_1 = 0$. Substitution of the given form of y_2 results in the differential equation $2x^{9/4} v'' + (4x^{7/4} + x^{5/4})v' = 0$. This equation is linear in the variable $w = v'$. An integrating factor is $\mu = e^{\int [2x^{-1/2} + 1/(2x)] dx} = \sqrt{x}\, e^{4\sqrt{x}}$. Rewrite the equation as $[\sqrt{x}\, e^{4\sqrt{x}}\, v']' = 0$, from which it follows that $v'(x) = c\, e^{-4\sqrt{x}}/\sqrt{x}$. Integrating, $v(x) = c_1 e^{-4\sqrt{x}} + c_2$ and as a result, $y_2(x) = c_1 x^{1/4} e^{-2\sqrt{x}} + c_2 x^{1/4} e^{2\sqrt{x}}$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x^{1/4} e^{-2\sqrt{x}}$.

31. Direct substitution verifies that $y_1(t) = e^{-\delta x^2/2}$ is a solution of the differential equation. Now set $y_2(x) = y_1(x)\, v(x)$. Substitution of y_2 into the equation results in $v'' - \delta x v' = 0$. This equation is linear in the variable $w = v'$. An integrating factor is $\mu = e^{-\delta x^2/2}$. Rewrite the equation as $[\, e^{-\delta x^2/2} v']' = 0$, from which it follows that $v'(x) = c_1\, e^{\delta x^2/2}$. Integrating, we obtain

$$v(x) = c_1 \int_0^x e^{\delta u^2/2} du + v(0).$$

Hence

$$y_2(x) = c_1 e^{-\delta x^2/2} \int_0^x e^{\delta u^2/2} du + c_2 e^{-\delta x^2/2}.$$

Setting $c_2 = 0$, we obtain a second independent solution.

33. After writing the differential equation in standard form, we have $p(t) = 3/t$. Based on Abel's identity, $W(y_1, y_2) = c_1 e^{-\int 3/t\, dt} = c_1 t^{-3}$. As shown in Problem 32, two solutions of a second order linear equation satisfy $(y_2/y_1)' = W(y_1, y_2)/y_1^2$. In the given problem, $y_1(t) = t^{-1}$. Hence $(t\, y_2)' = c_1 t^{-1}$. Integrating both sides of the equation, $y_2(t) = c_1 t^{-1} \ln t + c_2 t^{-1}$. Setting $c_1 = 1$ and $c_2 = 0$ we obtain $y_2(t) = t^{-1} \ln t$.

35. After writing the differential equation in standard form, we have $p(x) = -x/(x-1)$. Based on Abel's identity, $W(y_1, y_2) = c\, e^{\int x/(x-1)\, dx} = c\, e^x (x-1)$. Two solutions of a second order linear equation satisfy $(y_2/y_1)' = W(y_1, y_2)/y_1^2$. In the given problem, $y_1(x) = e^x$. Hence $(e^{-x}\, y_2)' = c\, e^{-x}(x-1)$. Integrating both

sides of the equation, $y_2(x) = c_1 x + c_2 e^x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x$.

36. Write the differential equation in standard form to find $p(x) = 1/x$. Based on Abel's identity, $W(y_1, y_2) = c\, e^{-\int 1/x\, dx} = c\, x^{-1}$. Two solutions of a second order linear differential equation satisfy $(y_2/y_1)' = W(y_1, y_2)/y_1^2$. In the given problem, $y_1(x) = x^{-1/2} \sin x$. Hence

$$\left(\frac{\sqrt{x}}{\sin x}\, y_2\right)' = c\, \frac{1}{\sin^2 x}\,.$$

Integrating both sides of the equation, $y_2(x) = c_1 x^{-1/2} \cos x + c_2 x^{-1/2} \sin x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x^{-1/2} \cos x$.

38.(a) The characteristic equation is $ar^2 + c = 0$. If $a, c > 0$, then the roots are $r = \pm i \sqrt{c/a}$. The general solution is

$$y(t) = c_1 \cos \sqrt{\frac{c}{a}}\, t + c_2 \sin \sqrt{\frac{c}{a}}\, t\,,$$

which is bounded.

(b) The characteristic equation is $ar^2 + br = 0$. The roots are $r = 0, -b/a$, and hence the general solution is $y(t) = c_1 + c_2 e^{-bt/a}$. Clearly, $y(t) \to c_1$. With the given initial conditions, $c_1 = y_0 + (a/b) y_0'$.

39. Note that $2 \cos t \sin t = \sin 2t$. Then $1 - k \cos t \sin t = 1 - (k/2) \sin 2t$. Now if $0 < k < 2$, then $(k/2) \sin 2t < |\sin 2t|$ and $-(k/2) \sin 2t > -|\sin 2t|$. Hence

$$1 - k \cos t \sin t = 1 - \frac{k}{2} \sin 2t > 1 - |\sin 2t| \ge 0\,.$$

40. The equation transforms into $y'' - 4y' + 4y = 0$. We obtain a double root $r = 2$. The solution is $y = c_1 e^{2x} + c_2 x e^{2x} = c_1 e^{2 \ln t} + c_2 \ln t e^{2 \ln t} = c_1 t^2 + c_2 t^2 \ln t$.

42. The equation transforms into $y'' - 7y'/2 + 5y/2 = 0$. The characteristic roots are $r = 1,\, 5/2$, so the solution is $y = c_1 e^x + c_2 e^{5x/2} = c_1 e^{\ln t} + c_2 e^{5 \ln t/2} = c_1 t + c_2 t^{5/2}$.

43. The equation transforms into $y'' + 2y' + y = 0$. We get a double root $r = -1$. The solution is $y = c_1 e^{-x} + c_2 x e^{-x} = c_1 e^{-\ln t} + c_2 \ln t e^{-\ln t} = c_1 t^{-1} + c_2 t^{-1} \ln t$.

44. The equation transforms into $y'' - 3y' + 9y/4 = 0$. We obtain the double root $r = 3/2$. The solution is $y = c_1 e^{3x/2} + c_2 x e^{3x/2} = c_1 e^{3 \ln t/2} + c_2 \ln t e^{3 \ln t/2} = c_1 t^{3/2} + c_2 t^{3/2} \ln t$.