[redacted]

3.3

2. $e^{2-3i} = e^2 e^{-3i} = e^2(\cos 3 - i \sin 3)$.

3. $e^{i\pi} = \cos \pi + i \sin \pi = -1$.

4. $e^{2-(\pi/2)\,i} = e^2(\cos(\pi/2) - i \sin(\pi/2)) = -e^2\, i$.

6. $\pi^{-1+2i} = e^{(-1+2i)\ln \pi} = e^{-\ln \pi} e^{2 \ln \pi\, i} = (\cos(2 \ln \pi) + i \sin(2 \ln \pi))/\pi$.

8. The characteristic equation is $r^2 - 2r + 6 = 0$, with roots $r = 1 \pm i\sqrt{5}$. Hence the general solution is $y = c_1 e^t \cos \sqrt{5}\, t + c_2\, e^t \sin \sqrt{5}\, t$.

9. The characteristic equation is $r^2 + 2r - 8 = 0$, with roots $r = -4\,, 2$. The roots are real and different, hence the general solution is $y = c_1 e^{-4t} + c_2\, e^{2t}$.

10. The characteristic equation is $r^2 + 2r + 2 = 0$, with roots $r = -1 \pm i$. Hence the general solution is $y = c_1 e^{-t} \cos t + c_2\, e^{-t} \sin t$.

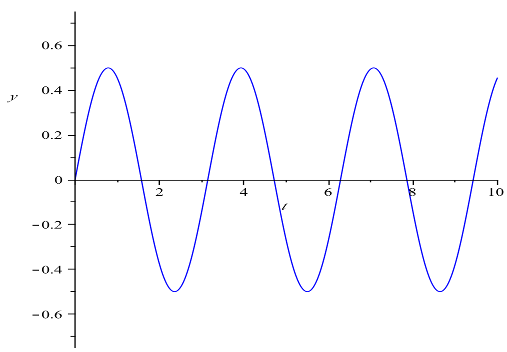
12. The characteristic equation is $4r^2 + 9 = 0$, with roots $r = \pm(3/2)\,i$. Hence the general solution is $y = c_1 \cos(3t/2) + c_2 \sin(3t/2)$.

13. The characteristic equation is $r^2 + 2r + 1.25 = 0$, with roots $r = -1 \pm i/2$. Hence the general solution is $y = c_1 e^{-t} \cos(t/2) + c_2\, e^{-t} \sin(t/2)$.

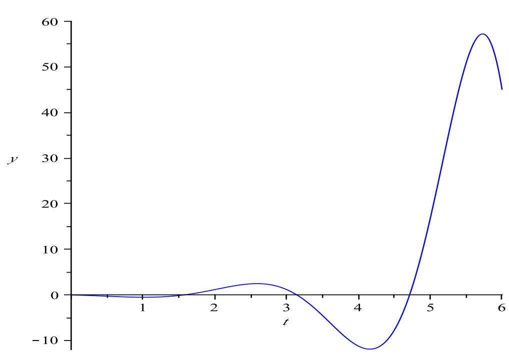
15. The characteristic equation is $r^2 + r + 1.25 = 0$, with roots $r = -(1/2) \pm i$. Hence the general solution is $y = c_1 e^{-t/2} \cos t + c_2\, e^{-t/2} \sin t$.

16. The characteristic equation is $r^2 + 4r + 6.25 = 0$, with roots $r = -2 \pm (3/2)\,i$. Hence the general solution is $y = c_1 e^{-2t} \cos(3t/2) + c_2\, e^{-2t} \sin(3t/2)$.

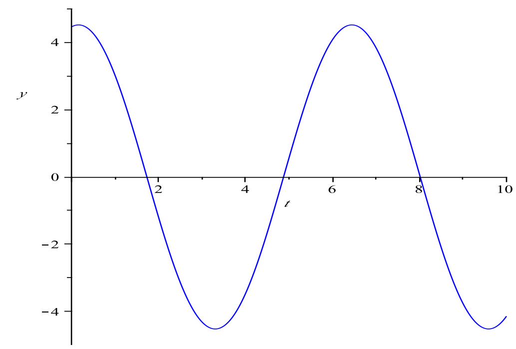
17. The characteristic equation is $r^2 + 4 = 0$, with roots $r = \pm 2i$. Hence the general solution is $y = c_1 \cos 2t + c_2 \sin 2t$. Now $y' = -2c_1 \sin 2t + 2c_2 \cos 2t$. Based on the first condition, $y(0) = 0$, we require that $c_1 = 0$. In order to satisfy the condition $y'(0) = 1$, we find that $2c_2 = 1$. The constants are $c_1 = 0$ and $c_2 = 1/2$. Hence the specific solution is $y(t) = \sin 2t\,/2$. The solution is periodic.



19. The characteristic equation is $r^2 - 2r + 5 = 0$, with roots $r = 1 \pm 2i$. Hence the general solution is $y = c_1 e^t \cos 2t + c_2\, e^t \sin 2t$. Based on the initial condition $y(\pi/2) = 0$, we require that $c_1 = 0$. It follows that $y = c_2\, e^t \sin 2t$, and so the first derivative is $y' = c_2\, e^t \sin 2t + 2c_2\, e^t \cos 2t$. In order to satisfy the condition $y'(\pi/2) = 2$, we find that $-2e^{\pi/2}c_2 = 2$. Hence we have $c_2 = -e^{-\pi/2}$. Therefore the specific solution is $y(t) = -e^{t-\pi/2} \sin 2t$. The solution oscillates with an exponentially growing amplitude.



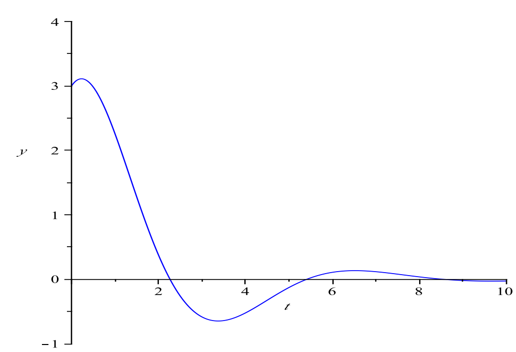
20. The characteristic equation is $r^2+1=0$, with roots $r=\pm i$. Hence the general solution is $y=c_1\cos t+c_2\sin t$. Its derivative is $y'=-c_1\sin t+c_2\cos t$. Based on the first condition, $y(\pi/3)=2$, we require that $c_1+\sqrt{3}\,c_2=4$. In order to satisfy the condition $y'(\pi/3)=-4$, we find that $-\sqrt{3}\,c_1+c_2=-8$. Solving these for the constants, $c_1=1+2\sqrt{3}$ and $c_2=\sqrt{3}-2$. Hence the specific solution is a steady oscillation, given by $y(t)=(1+2\sqrt{3})\cos t+(\sqrt{3}-2)\sin t$.



21. From Problem 15, the general solution is $y=c_1e^{-t/2}\cos t+c_2\,e^{-t/2}\sin t$. Invoking the first initial condition, $y(0)=3$, which implies that $c_1=3$. Substituting, it follows that $y=3e^{-t/2}\cos t+c_2\,e^{-t/2}\sin t$, and so the first derivative is

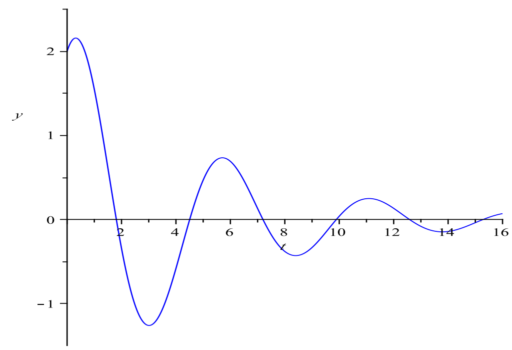
$$y'=-\frac{3}{2}e^{-t/2}\cos t-3e^{-t/2}\sin t+c_2\,e^{-t/2}\cos t-\frac{c_2}{2}e^{-t/2}\sin t\,.$$

Invoking the initial condition, $y'(0)=1$, we find that $-3/2+c_2=1$, and so $c_2=5/2$. Hence the specific solution is $y(t)=3e^{-t/2}\cos t+(5/2)\,e^{-t/2}\sin t$. It oscillates with an exponentially decreasing amplitude.



24.(a) The characteristic equation is $5r^2+2r+7=0$, with roots $r=-(1\pm i\sqrt{34})/5$. The solution is $u=c_1e^{-t/5}\cos\sqrt{34}t/5+c_2e^{-t/5}\sin\sqrt{34}t/5$. Invoking the given initial conditions, we obtain the equations for the coefficients: $c_1=2$, $-2+\sqrt{34}\,c_2=5$. That is, $c_1=2$, $c_2=7/\sqrt{34}$. Hence the specific solution is

$$u(t)=2e^{-t/5}\cos\frac{\sqrt{34}}{5}t+\frac{7}{\sqrt{34}}e^{-t/5}\sin\frac{\sqrt{34}}{5}t\,.$$



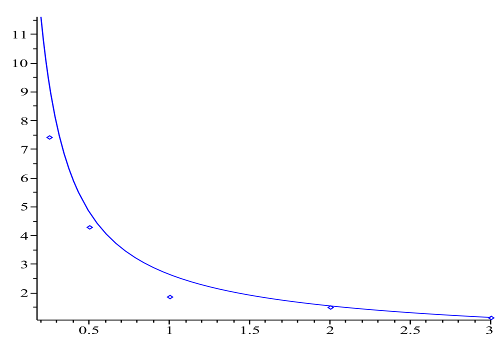
(b) Based on the graph of $u(t)$, T is in the interval $14 < t < 16$. A numerical solution on that interval yields $T \approx 14.5115$.

26.(a) The characteristic equation is $r^2 + 2ar + (a^2 + 1) = 0$, with roots $r = -a \pm i$. Hence the general solution is $y(t) = c_1 e^{-at} \cos t + c_2 e^{-at} \sin t$. Based on the initial conditions, we find that $c_1 = 1$ and $c_2 = a$. Therefore the specific solution is given by $y(t) = e^{-at} \cos t + a\, e^{-at} \sin t = \sqrt{1 + a^2}\, e^{-at} \cos(t - \phi)$, in which $\phi = \arctan(a)$.

(b) For estimation, note that $|y(t)| \leq \sqrt{1 + a^2}\, e^{-at}$. Now consider the inequality $\sqrt{1 + a^2}\, e^{-at} \leq 1/10$. The inequality holds for $t \geq (1/a) \ln(10\sqrt{1 + a^2})$. Therefore $T \leq (1/a) \ln(10\sqrt{1 + a^2})$. Setting $a = 1$, the numerical value is $T \approx 1.8763$.

(c) Similarly, $T_{1/4} \approx 7.4284$, $T_{1/2} \approx 4.3003$, $T_2 \approx 1.5116$.

(d)



Note that the estimates T_a approach the graph of $(1/a) \ln(10\sqrt{1 + a^2})$ as a gets large.

27. Direct calculation gives the result. On the other hand, it was shown in Problem 3.2.37 that $W(f\,g\,, f\,h) = f^2 W(g\,, h)$. Hence $W(e^{\lambda t} \cos \mu t\,, e^{\lambda t} \sin \mu t) = e^{2\lambda t}\, W(\cos \mu t\,, \sin \mu t) = e^{2\lambda t}\, [\cos \mu t (\sin \mu t)' - (\cos \mu t)' \sin \mu t] = \mu\, e^{2\lambda t}$.

28.(a) Clearly, y_1 and y_2 are solutions. Also, $W(\cos t, \sin t) = \cos^2 t + \sin^2 t = 1$.

(b) $y' = i\,e^{it}$, $y'' = i^2\,e^{it} = -e^{it}$. Evidently, y is a solution and so $y = c_1 y_1 + c_2 y_2$.

(c) Setting $t = 0$, $1 = c_1 \cos 0 + c_2 \sin 0$, and $c_1 = 1$.

(d) Differentiating, $i\,e^{it} = c_2 \cos t$. Setting $t = 0$, $i = c_2 \cos 0$ and hence $c_2 = i$. Therefore $e^{it} = \cos t + i \sin t$.

29. Euler's formula is $e^{it} = \cos t + i \sin t$. It follows that $e^{-it} = \cos t - i \sin t$. Adding these equation, $e^{it} + e^{-it} = 2 \cos t$. Subtracting the two equations results in $e^{it} - e^{-it} = 2i \sin t$.

30. Let $r_1 = \lambda_1 + i\mu_1$, and $r_2 = \lambda_2 + i\mu_2$. Then

$$\begin{aligned} e^{(r_1+r_2)t} &= e^{(\lambda_1+\lambda_2)t + i(\mu_1+\mu_2)t} = e^{(\lambda_1+\lambda_2)t}\left[\cos(\mu_1+\mu_2)t + i\,\sin(\mu_1+\mu_2)t\right] = \\ &= e^{(\lambda_1+\lambda_2)t}\left[(\cos\,\mu_1 t + i \sin\,\mu_1 t)(\cos\,\mu_2 t + i \sin\,\mu_2 t)\right] = \\ &= e^{\lambda_1 t}(\cos\,\mu_1 t + i \sin\,\mu_1 t)\cdot e^{\lambda_2 t}(\cos\,\mu_1 t + i \sin\,\mu_1 t) = e^{r_1 t}\,e^{r_2 t}. \end{aligned}$$

Hence $e^{(r_1+r_2)t} = e^{r_1 t}\,e^{r_2 t}$.

32. Clearly, $u' = \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t = e^{\lambda t}(\lambda\cos\mu t - \mu\sin\mu t)$ and then $u'' = \lambda e^{\lambda t}(\lambda\cos\mu t - \mu\sin\mu t) + e^{\lambda t}(-\lambda\mu\sin\mu t - \mu^2\cos\mu t)$. Plugging these into the differential equation, dividing by $e^{\lambda t} \neq 0$ and arranging the sine and cosine terms we obtain that the identity to prove is

$$(a(\lambda^2 - \mu^2) + b\lambda + c)\cos\mu t + (-2\lambda\mu a - b\mu)\sin\mu t = 0.$$

We know that $\lambda \pm i\mu$ solves the characteristic equation $ar^2 + br + c = 0$, so $a(\lambda - i\mu)^2 + b(\lambda - i\mu) + c = a(\lambda^2 - \mu^2) + b\lambda + c + i(-2\lambda\mu a - \mu b) = 0$. If this complex number is zero, then both the real and imaginary parts of it are zero, but those are the coefficients of $\cos\mu t$ and $\sin\mu t$ in the above identity, which proves that $au'' + bu' + cu = 0$. The solution for v is analogous.

35. The equation transforms into $y'' + y = 0$. The characteristic roots are $r = \pm i$. The solution is $y = c_1\cos(x) + c_2\sin(x) = c_1\cos(\ln t) + c_2\sin(\ln t)$.

37. The equation transforms into $y'' + 2y' + 1.25y = 0$. The characteristic roots are $r = -1 \pm i/2$. The solution is

$$y = c_1 e^{-x}\cos(x/2) + c_2 e^{-x}\sin(x/2) = c_1\frac{\cos(\frac{1}{2}\ln t)}{t} + c_2\frac{\sin(\frac{1}{2}\ln t)}{t}.$$

38. The equation transforms into $y'' - 5y' - 6y = 0$. The characteristic roots are $r = -1,\,6$. The solution is $y = c_1 e^{-x} + c_2 e^{6x} = c_1 e^{-\ln t} + c_2 e^{6\ln t} = c_1/t + c_2 t^6$.

39. The equation transforms into $y'' - 5y' + 6y = 0$. The characteristic roots are $r = 2,\,3$. The solution is $y = c_1 e^{2x} + c_2 e^{3x} = c_1 e^{2\ln t} + c_2 e^{3\ln t} = c_1 t^2 + c_2 t^3$.

41. The equation transforms into $y'' + 2y' - 3y = 0$. The characteristic roots are $r = 1,\,-3$. The solution is $y = c_1 e^{x} + c_2 e^{-3x} = c_1 e^{\ln t} + c_2 e^{-3\ln t} = c_1 t + c_2/t^3$.

42. The equation transforms into $y'' + 6y' + 10y = 0$. The characteristic roots are $r = -3 \pm i$. The solution is

$$y = c_1 e^{-3x}\cos(x) + c_2 e^{-3x}\sin(x) = c_1 \frac{1}{t^3}\cos(\ln t) + c_2 \frac{1}{t^3}\sin(\ln t).$$

43.(a) By the chain rule, $y'(x) = (dy/dx)x'$. In general, $dz/dt = (dz/dx)(dx/dt)$. Setting $z = (dy/dt)$, we have

$$\frac{d^2y}{dt^2} = \frac{dz}{dx}\frac{dx}{dt} = \frac{d}{dx}\left[\frac{dy}{dx}\frac{dx}{dt}\right]\frac{dx}{dt} = \left[\frac{d^2y}{dx^2}\frac{dx}{dt}\right]\frac{dx}{dt} + \frac{dy}{dx}\frac{d}{dx}\left[\frac{dx}{dt}\right]\frac{dx}{dt}.$$

However,

$$\frac{d}{dx}\left[\frac{dx}{dt}\right]\frac{dx}{dt} = \left[\frac{d^2x}{dt^2}\right]\frac{dt}{dx}\cdot\frac{dx}{dt} = \frac{d^2x}{dt^2}.$$

Hence

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left[\frac{dx}{dt}\right]^2 + \frac{dy}{dx}\frac{d^2x}{dt^2}.$$

(b) Substituting the results in part (a) into the general differential equation, $y'' + p(t)y' + q(t)y = 0$, we find that

$$\frac{d^2y}{dx^2}\left[\frac{dx}{dt}\right]^2 + \frac{dy}{dx}\frac{d^2x}{dt^2} + p(t)\frac{dy}{dx}\frac{dx}{dt} + q(t)y = 0\,.$$

Collecting the terms,

$$\left[\frac{dx}{dt}\right]^2\frac{d^2y}{dx^2} + \left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right]\frac{dy}{dx} + q(t)y = 0\,.$$

(c) Assuming $(dx/dt)^2 = k\,q(t)$, and $q(t) > 0$, we find that $dx/dt = \sqrt{k\,q(t)}$, which can be integrated. That is, $x = u(t) = \int \sqrt{k\,q(t)}\,dt = \int \sqrt{q(t)}\,dt$, since $k = 1$.

(d) Let $k = 1$. It follows that $d^2x/dt^2 + p(t)dx/dt = du/dt + p(t)u(t) = q'/2\sqrt{q} + p\sqrt{q}$. Hence

$$\left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right]\Big/\left[\frac{dx}{dt}\right]^2 = \frac{q'(t) + 2p(t)q(t)}{2\,[q(t)]^{3/2}}\,.$$

As long as $dx/dt \neq 0$, the differential equation can be expressed as

$$\frac{d^2y}{dx^2} + \left[\frac{q'(t) + 2p(t)q(t)}{2\,[q(t)]^{3/2}}\right]\frac{dy}{dx} + y = 0.$$

For the case $q(t) < 0$, write $q(t) = -[-q(t)]$, and set $(dx/dt)^2 = -q(t)$.

45. $p(t) = 3t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2\,[q(t)]^{3/2}} = \frac{1 + 3t^2}{t^2}\,.$$

The ratio is not constant, and therefore the equation cannot be transformed.

46. $p(t) = t - 1/t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2\left[q(t)\right]^{3/2}} = 1\,.$$

The ratio is constant, and therefore the equation can be transformed. From Problem 43, the transformed equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0\,.$$

Based on the methods in this section, the characteristic equation is $r^2 + r + 1 = 0$, with roots $r = (-1 \pm i\sqrt{3})/2$. The general solution is $y(x) = c_1 e^{-x/2}\cos\sqrt{3}\,x/2 + c_2\, e^{-x/2}\sin\sqrt{3}\,x/2$. Since $x = t^2/2$, the solution in the original variable t is

$$y(t) = e^{-t^2/4}\left[c_1\cos(\sqrt{3}\,t^2/4) + c_2\,\sin(\sqrt{3}\,t^2/4)\right]\,.$$

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