CHAPTER

3

Second Order Linear Equations

3.1

1. Let $y = e^{rt}$, so that $y' = r\,e^{rt}$ and $y'' = r^2\,e^{rt}$. Direct substitution into the differential equation yields $(r^2 + 2r - 3)e^{rt} = 0$. Canceling the exponential, the characteristic equation is $r^2 + 2r - 3 = 0$. The roots of the equation are $r = -3\,, 1$. Hence the general solution is $y = c_1 e^t + c_2 e^{-3t}$.

2. Let $y = e^{rt}$. Substitution of the assumed solution results in the characteristic equation $r^2 + 3r + 2 = 0$. The roots of the equation are $r = -2\,, -1$. Hence the general solution is $y = c_1 e^{-t} + c_2 e^{-2t}$.

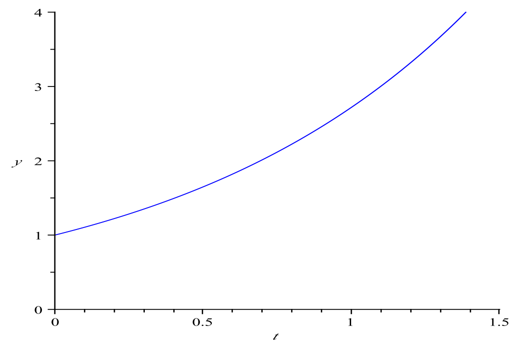
4. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $2r^2 - 3r + 1 = 0$. The roots of the equation are $r = 1/2\,, 1$. Hence the general solution is $y = c_1 e^{t/2} + c_2 e^t$.

6. The characteristic equation is $4r^2 - 9 = 0$, with roots $r = \pm 3/2$. Therefore the general solution is $y = c_1 e^{-3t/2} + c_2 e^{3t/2}$.

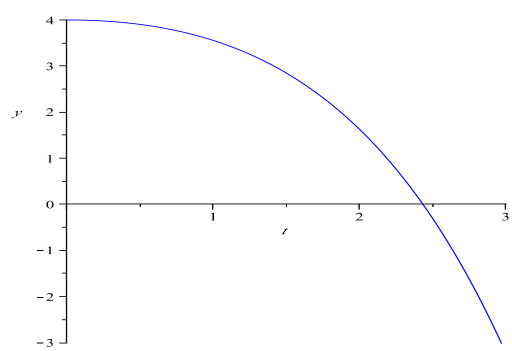
8. The characteristic equation is $r^2 - 2r - 2 = 0$, with roots $r = 1 \pm \sqrt{3}$. Hence the general solution is $y = c_1 e^{(1-\sqrt{3})t} + c_2 e^{(1+\sqrt{3})t}$.

9. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + r - 2 = 0$. The roots of the equation are $r = -2\,, 1$. Hence the general solution is $y = c_1 e^{-2t} + c_2 e^t$. Its derivative is $y' = -2c_1 e^{-2t} + c_2 e^t$. Based on the

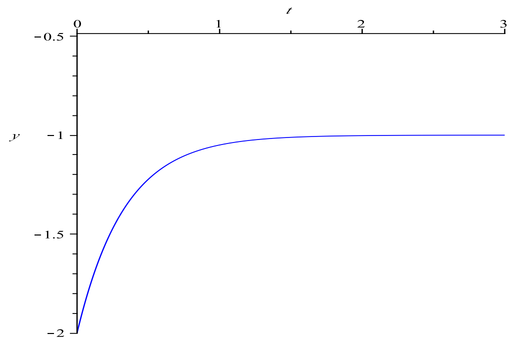
first condition, $y(0) = 1$, we require that $c_1 + c_2 = 1$. In order to satisfy $y'(0) = 1$, we find that $-2c_1 + c_2 = 1$. Solving for the constants, $c_1 = 0$ and $c_2 = 1$. Hence the specific solution is $y(t) = e^t$. It clearly increases without bound as $t \to \infty$.



11. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $6r^2 - 5r + 1 = 0$. The roots of the equation are $r = 1/3\,, 1/2$. Hence the general solution is $y = c_1 e^{t/3} + c_2 e^{t/2}$. Its derivative is $y' = c_1 e^{t/3}/3 + c_2 e^{t/2}/2$. Based on the first condition, $y(0) = 1$, we require that $c_1 + c_2 = 4$. In order to satisfy the condition $y'(0) = 1$, we find that $c_1/3 + c_2/2 = 0$. Solving for the constants, $c_1 = 12$ and $c_2 = -8$. Hence the specific solution is $y(t) = 12\,e^{t/3} - 8\,e^{t/2}$. It clearly decreases without bound as $t \to \infty$.



12. The characteristic equation is $r^2 + 3r = 0$, with roots $r = -3\,,\ 0$. Therefore the general solution is $y = c_1 + c_2 e^{-3t}$, with derivative $y' = -3\,c_2 e^{-3t}$. In order to satisfy the initial conditions, we find that $c_1 + c_2 = -2$, and $-3\,c_2 = 3$. Hence the specific solution is $y(t) = -1 - e^{-3t}$. This converges to -1 as $t \to \infty$.



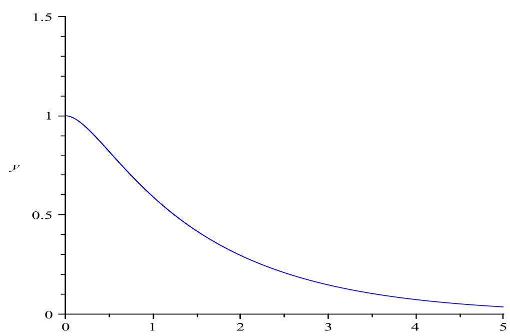
13. The characteristic equation is $r^2 + 5r + 3 = 0$, with roots $r = (-5 \pm \sqrt{13})/2$. The general solution is $y = c_1 e^{(-5-\sqrt{13})t/2} + c_2 e^{(-5+\sqrt{13})t/2}$, with derivative

$$y' = \frac{-5-\sqrt{13}}{2}\, c_1 e^{(-5-\sqrt{13})t/2} + \frac{-5+\sqrt{13}}{2}\, c_2 e^{(-5+\sqrt{13})t/2}\,.$$

In order to satisfy the initial conditions, we require that

$$c_1 + c_2 = 1 \quad \text{and} \quad \frac{-5-\sqrt{13}}{2}\, c_1 + \frac{-5+\sqrt{13}}{2}\, c_2 = 0.$$

Solving for the coefficients, $c_1 = (1 - 5/\sqrt{13})/2$ and $c_2 = (1 + 5/\sqrt{13})/2$. The solution clearly converges to 0 as $t \to \infty$.



14. The characteristic equation is $2r^2 + r - 4 = 0$, with roots $r = (-1 \pm \sqrt{33})/4$. The general solution is $y = c_1 e^{(-1-\sqrt{33})t/4} + c_2 e^{(-1+\sqrt{33})t/4}$, with derivative

$$y' = \frac{-1-\sqrt{33}}{4}\, c_1 e^{(-1-\sqrt{33})t/4} + \frac{-1+\sqrt{33}}{4}\, c_2 e^{(-1+\sqrt{33})t/4}\,.$$

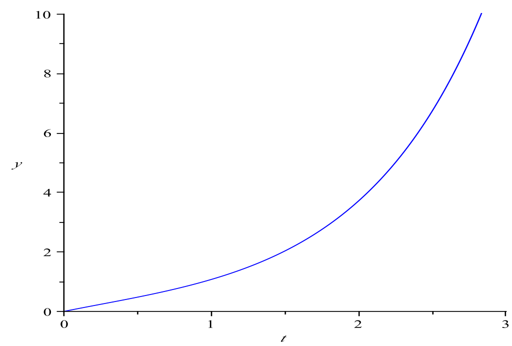
In order to satisfy the initial conditions, we require that

$$c_1 + c_2 = 0 \quad \text{and} \quad \frac{-1-\sqrt{33}}{4}\, c_1 + \frac{-1+\sqrt{33}}{4}\, c_2 = 1.$$

Solving for the coefficients, $c_1 = -2/\sqrt{33}$ and $c_2 = 2/\sqrt{33}$. The specific solution is

$$y(t) = -2\left[e^{(-1-\sqrt{33})t/4} - e^{(-1+\sqrt{33})t/4}\right]/\sqrt{33}\,.$$

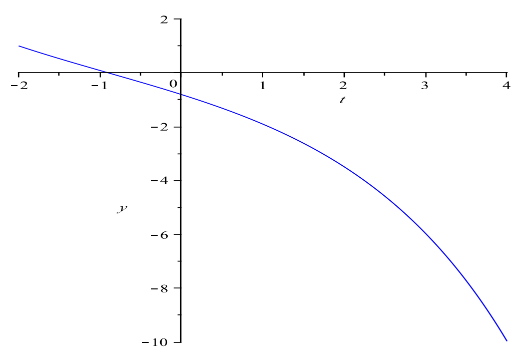
It clearly increases without bound as $t \to \infty$.



16. The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Therefore the general solution is $y = c_1 e^{-t/2} + c_2 e^{t/2}$. Since the initial conditions are specified at $t = -2$, is more convenient to write $y = d_1 e^{-(t+2)/2} + d_2 e^{(t+2)/2}$. The derivative is given by $y' = -\left[d_1 e^{-(t+2)/2}\right]/2 + \left[d_2 e^{(t+2)/2}\right]/2$. In order to satisfy the initial conditions, we find that $d_1 + d_2 = 1$, and $-d_1/2 + d_2/2 = -1$. Solving for the coefficients, $d_1 = 3/2$, and $d_2 = -1/2$. The specific solution is

$$y(t) = \frac{3}{2} e^{-(t+2)/2} - \frac{1}{2} e^{(t+2)/2} = \frac{3}{2e} e^{-t/2} - \frac{e}{2} e^{t/2}.$$

It clearly decreases without bound as $t \to \infty$.



18. An algebraic equation with roots -2 and $-1/2$ is $2r^2 + 5r + 2 = 0$. This is the characteristic equation for the differential equation $2y'' + 5y' + 2y = 0$.

20. The characteristic equation is $2r^2 - 3r + 1 = 0$, with roots $r = 1/2, 1$. Therefore the general solution is $y = c_1 e^{t/2} + c_2 e^t$, with derivative $y' = c_1 e^{t/2}/2 + c_2 e^t$. In order to satisfy the initial conditions, we require $c_1 + c_2 = 2$ and $c_1/2 + c_2 = 1/2$. Solving for the coefficients, $c_1 = 3$, and $c_2 = -1$. The specific solution is $y(t) = 3e^{t/2} - e^t$. To find the stationary point, set $y' = 3e^{t/2}/2 - e^t = 0$. There is a unique solution, with $t_1 = \ln(9/4)$. The maximum value is then $y(t_1) = 9/4$. To find the x-intercept, solve the equation $3e^{t/2} - e^t = 0$. The solution is readily found to be $t_2 = \ln 9 \approx 2.1972$.

22. The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Hence the general solution is $y = c_1 e^{-t/2} + c_2 e^{t/2}$ and $y' = -c_1 e^{-t/2}/2 + c_2 e^{t/2}/2$. Invoking the initial conditions, we require that $c_1 + c_2 = 2$ and $-c_1 + c_2 = 2\beta$. The specific solution is $y(t) = (1-\beta)e^{-t/2} + (1+\beta)e^{t/2}$. Based on the form of the solution, it is evident that as $t \to \infty$, $y(t) \to 0$ as long as $\beta = -1$.

23. The characteristic equation is $r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0$. Examining the coefficients, the roots are $r = \alpha$, $\alpha - 1$. Hence the general solution of the differential equation is $y(t) = c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t}$. Assuming $\alpha \in \mathbb{R}$, all solutions will tend to zero as long as $\alpha < 0$. On the other hand, all solutions will become unbounded as long as $\alpha - 1 > 0$, that is, $\alpha > 1$.

26.(a) The characteristic roots are $r = -3, -2$. The solution of the initial value problem is $y(t) = (6+\beta)e^{-2t} - (4+\beta)e^{-3t}$.

(b) The maximum point has coordinates $t_0 = \ln\left[(3(4+\beta))/(2(6+\beta))\right]$, $y_0 = 4(6+\beta)^3/(27(4+\beta)^2)$.

(c) $y_0 = 4(6+\beta)^3/(27(4+\beta)^2) \geq 4$, as long as $\beta \geq 6 + 6\sqrt{3}$.

(d) $\lim_{\beta\to\infty} t_0 = \ln(3/2)$, $\lim_{\beta\to\infty} y_0 = \infty$.

27.(a) Assuming that y is a constant, the differential equation reduces to $c\,y = d$. Hence the only equilibrium solution is $y = d/c$.

(b) Setting $y = Y + d/c$, substitution into the differential equation results in the equation $a\,Y'' + b\,Y' + c(Y + d/c) = d$. The equation satisfied by Y is $a\,Y'' + b\,Y' + cY = 0$.

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