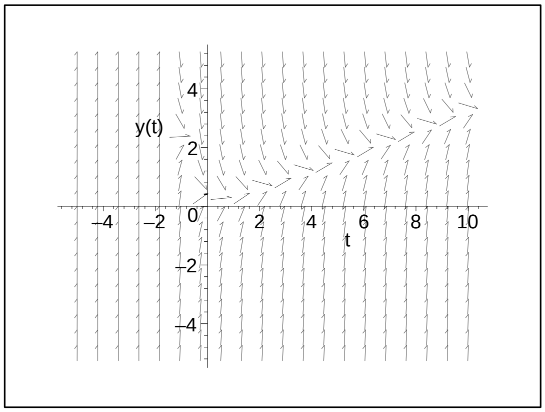
Chapter 2
Section 2.1

1.

(a)



(b) All solutions seem to converge to an increasing function as $t \to \infty$.

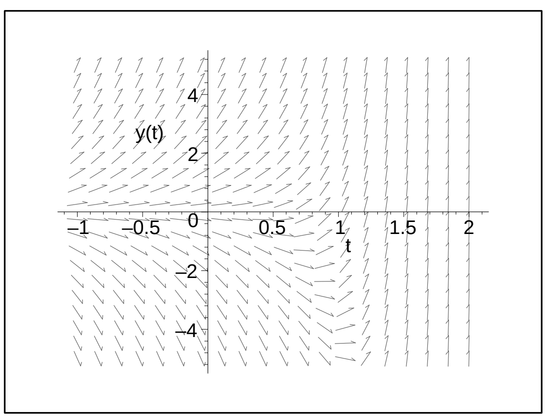
(c) The integrating factor is $\mu(t) = e^{3t}$. Then

$$e^{3t}y' + 3e^{3t}y = e^{3t}(t + e^{-2t}) \implies (e^{3t}y)' = te^{3t} + e^t$$
$$\implies e^{3t}y = \int (te^{3t} + e^t)\,dt = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + e^t + c$$
$$\implies y = \frac{t}{3} - \frac{1}{9} + e^{-2t} + ce^{-3t}.$$

We conclude that y is asymptotic to $t/3 - 1/9$ as $t \to \infty$.

2.

(a)



(b) All slopes eventually become positive, so all solutions will eventually increase without bound.

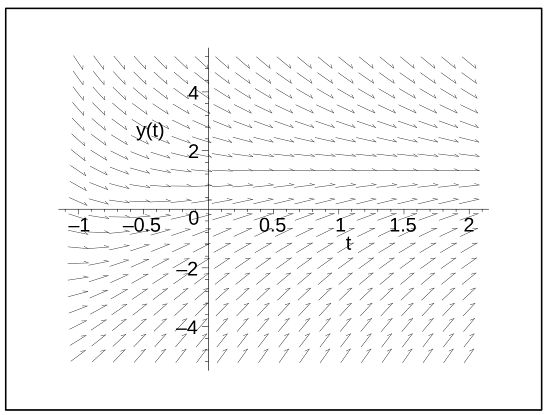
(c) The integrating factor is $\mu(t) = e^{-2t}$. Then

$$\begin{aligned} &e^{-2t}y' - 2e^{-2t}y = e^{-2t}(t^2e^{2t}) \implies (e^{-2t}y)' = t^2 \\ &\implies e^{-2t}y = \int t^2\,dt = \frac{t^3}{3} + c \\ &\implies y = \frac{t^3}{3}e^{2t} + ce^{2t}. \end{aligned}$$

We conclude that y increases exponentially as $t \to \infty$.

3.

(a)



(b) All solutions appear to converge to the function $y(t) = 1$.

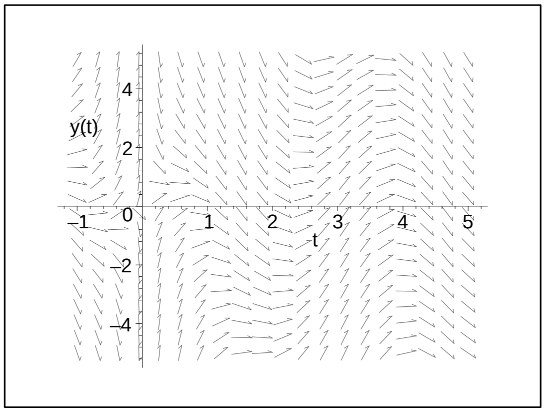
(c) The integrating factor is $\mu(t) = e^t$. Therefore,

$$\begin{aligned} &e^t y' + e^t y = t + e^t \implies (e^t y)' = t + e^t \\ &\implies e^t y = \int (t + e^t)\,dt = \frac{t^2}{2} + e^t + c \\ &\implies y = \frac{t^2}{2}e^{-t} + 1 + ce^{-t}. \end{aligned}$$

Therefore, we conclude that $y \to 1$ as $t \to \infty$.

4.

(a)



(b) The solutions eventually become oscillatory.

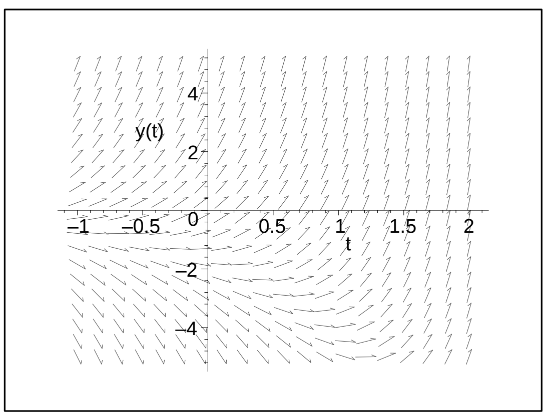
(c) The integrating factor is $\mu(t) = t$. Therefore,

$$\begin{aligned} ty' + y = 3t\cos(2t) &\implies (ty)' = 3t\cos(2t) \\ \implies ty = \int 3t\cos(2t)\,dt &= \frac{3}{4}\cos(2t) + \frac{3}{2}t\sin(2t) + c \\ \implies y = +\frac{3\cos 2t}{4t} &+ \frac{3\sin 2t}{2} + \frac{c}{t}. \end{aligned}$$

We conclude that y is asymptotic to $(3\sin 2t)/2$ as $t \to \infty$.

5.

(a)



(b) All slopes eventually become positive so all solutions eventually increase without bound.

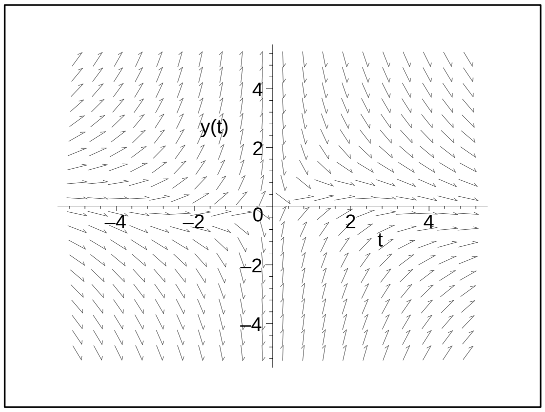
(c) The integrating factor is $\mu(t) = e^{-2t}$. Therefore,

$$\begin{aligned} e^{-2t}y' - 2e^{-2t}y = 3e^{-t} &\implies (e^{-2t}y)' = 3e^{-t} \\ \implies e^{-2t}y = \int 3e^{-t}\,dt &= -3e^{-t} + c \\ \implies y = -3e^{t} + ce^{2t}. \end{aligned}$$

We conclude that y increases exponentially as $t \to \infty$.

6.

(a)



(b) For $t > 0$, all solutions seem to eventually converge to the function $y = 0$.

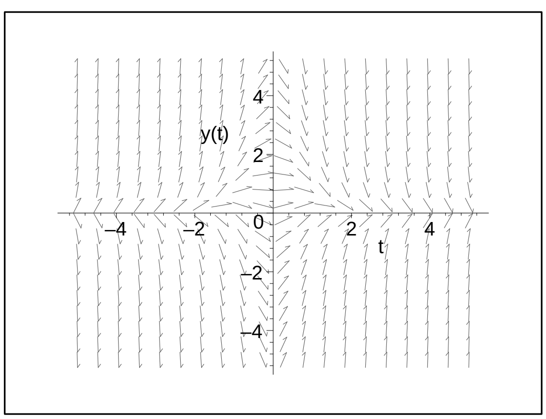
(c) The integrating factor is $\mu(t) = t^2$. Therefore,

$$t^2 y' + 2ty = t\sin(t) \implies (t^2 y)' = t\sin(t)$$
$$\implies t^2 y = \int t\sin(t)\,dt = \sin(t) - t\cos(t) + c$$
$$\implies y = \frac{\sin t - t\cos t + c}{t^2}.$$

We conclude that $y \to 0$ as $t \to \infty$.

7.

(a)

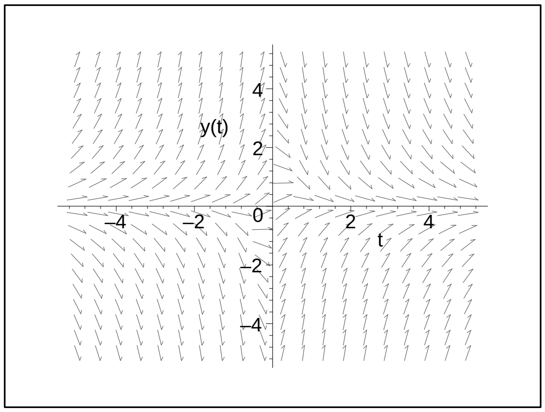


(b) For $t > 0$, all solutions seem to eventually converge to the function $y = 0$.

(c) The integrating factor is $\mu(t) = e^{t^2}$. Therefore, using the techniques shown above, we see that $y(t) = t^2 e^{-t^2} + ce^{-t^2}$. We conclude that $y \to 0$ as $t \to \infty$.

8.

(a)



(b) For $t > 0$, all solutions seem to eventually converge to the function $y = 0$.

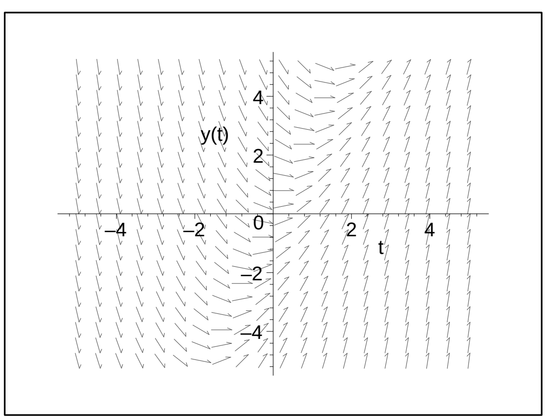
(c) The integrating factor is $\mu(t) = (1+t^2)^2$. Then

$$\begin{aligned}(1+t^2)^2 y' + 4t(1+t^2)y &= \frac{1}{1+t^2} \\ \implies ((1+t^2)^2 y) &= \int \frac{1}{1+t^2}\,dt \\ \implies y = (\tan^{-1}(t) + c)/(1+t^2)^2.\end{aligned}$$

We conclude that $y \to 0$ as $t \to \infty$.

9.

(a)



(b) All slopes eventually become positive. Therefore, all solutions will increase without bound.

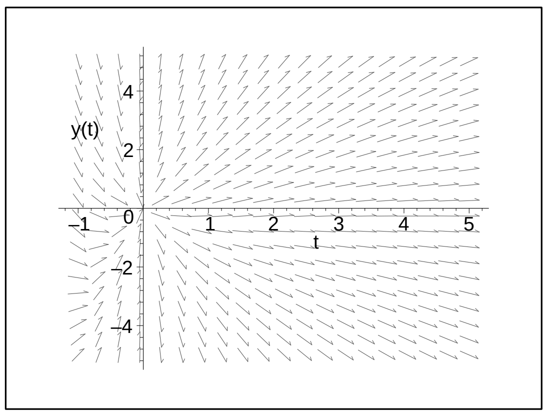
(c) The integrating factor is $\mu(t) = e^{t/2}$. Therefore,

$$2e^{t/2}y' + e^{t/2}y = 3te^{t/2} \qquad \Longrightarrow 2e^{t/2}y = \int 3te^{t/2}\,dt = 6te^{t/2} - 12e^{t/2} + c$$
$$\Longrightarrow y = 3t - 6 + ce^{-t/2}.$$

We conclude that $y \to 3t - 6$ as $t \to \infty$.

10.

(a)



(b) For $y > 0$, the slopes are all positive, and, therefore, the corresponding solutions increase without bound. For $y < 0$ almost all solutions have negative slope and therefore decrease without bound.

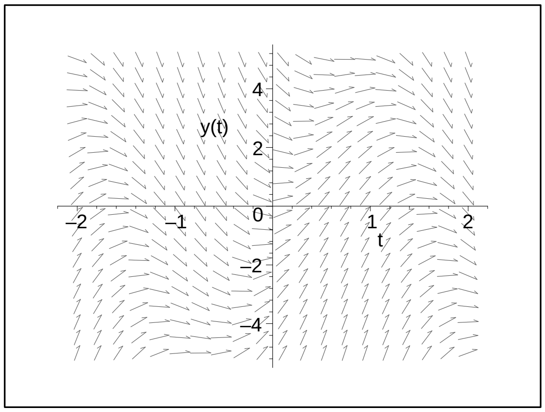
(c) By dividing the equation by t, we see that the integrating factor is $\mu(t) = 1/t$. Therefore,

$$y'/t - y/t^2 = te^{-t} \implies (y/t)' = te^{-t}$$
$$\implies \frac{y}{t} = \int te^{-t}\,dt = -te^{-t} - e^{-t} + c$$
$$\implies y = -t^2e^{-t} - te^{-t} + ct.$$

We conclude that $y \to \infty$ if $c > 0$, $y \to -\infty$ if $c < 0$ and $y \to 0$ if $c = 0$.

11.

(a)



(b) The solution appears to be oscillatory.

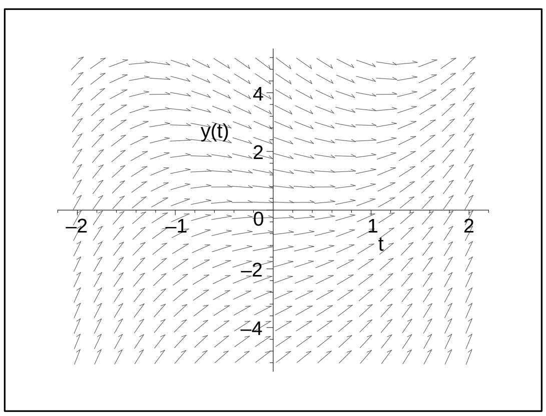
(c) The integrating factor is $\mu(t) = e^t$. Therefore,

$$e^t y' + e^t y = 5e^t \sin(2t) \implies (e^t y)' = 5e^t \sin(2t)$$
$$\implies e^t y = \int 5e^t \sin(2t)\, dt = -2e^t \cos(2t) + e^t \sin(2t) + c \quad \implies y = -2\cos(2t) + \sin(2t) + ce^{-t}.$$

We conclude that $y \to \sin(2t) - 2\cos(2t)$ as $t \to \infty$.

12.

(a)



(b) All slopes are eventually positive. Therefore, all solutions increase without bound.

(c) The integrating factor is $\mu(t) = e^{t/}$. Therefore,

$$2e^{t/2}y' + e^{t/2}y = 3t^2 e^{t/2} \implies (2e^{t/2}y)' = 3t^2 e^{t/2}$$
$$\implies 2e^{t/2}y = \int 3t^2 e^{t/2}\, dt = 6t^2 e^{t/2} - 24te^{t/2} + 48e^{t/2} + c$$
$$\implies y = 3t^2 - 12t + 24 + ce^{-t/2}.$$

We conclude that y is asymptotic to $3t^2 - 12t + 24$ as $t \to \infty$.

13. The integrating factor is $\mu(t) = e^{-t}$. Therefore,

$$(e^{-t}y)' = 2te^t \implies y = e^t \int 2te^t\,dt = 2te^{2t} - 2e^{2t} + ce^t.$$

The initial condition $y(0) = 1$ implies $-2 + c = 1$. Therefore, $c = 3$ and $y = 3e^t + 2(t-1)e^{2t}$

14. The integrating factor is $\mu(t) = e^{2t}$. Therefore,

$$(e^{2t}y)' = t \implies y = e^{-2t} \int t\,dt = \frac{t^2}{2}e^{-2t} + ce^{-2t}.$$

The initial condition $y(1) = 0$ implies $e^{-2t}/2 + ce^{-2t} = 0$. Therefore, $c = -1/2$, and $y = (t^2 - 1)e^{-2t}/2$

15. Dividing the equation by t, we see that the integrating factor is $\mu(t) = t^2$. Therefore,

$$(t^2y)' = t^3 - t^2 + t \implies y = t^{-2} \int (t^3 - t^2 + t)\,dt = \left(\frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{c}{t^2}\right).$$

The initial condition $y(1) = 1/2$ implies $c = 1/12$, and $y = (3t^4 - 4t^3 + 6t^2 + 1)/12t^2$.

16. The integrating factor is $\mu(t) = t^2$. Therefore,

$$(t^2y)' = \cos(t) \implies y = t^{-2} \int \cos(t)\,dt = t^{-2}(\sin(t) + c).$$

The initial condition $y(\pi) = 0$ implies $c = 0$ and $y = (\sin t)/t^2$

17. The integrating factor is $\mu(t) = e^{-2t}$. Therefore,

$$(e^{-2t}y)' = 1 \implies y = e^{2t} \int 1\,dt = e^{2t}(t + c).$$

The initial condition $y(0) = 2$ implies $c = 2$ and $y = (t+2)e^{2t}$.

18. After dividing by t, we see that the integrating factor is $\mu(t) = t^2$. Therefore,

$$(t^2y)' = 1 \implies y = t^{-2} \int t\sin(t)\,dt = t^{-2}(\sin(t) - t\cos(t) + c).$$

The initial condition $y(\pi/2) = 1$ implies $c = (\pi^2/4) - 1$ and $y = t^{-2}[(\pi^2/4) - 1 - t\cos t + \sin t]$.

19. After dividing by t^3, we see that the integrating factor is $\mu(t) = t^4$. Therefore,

$$(t^4y)' = te^{-t} \implies y = t^{-4} \int te^{-t}\,dt = t^{-4}(-te^{-t} - e^{-t} + c).$$

The initial condition $y(-1) = 0$ implies $c = 0$ and $y = -(1+t)e^{-t}/t^4, \quad t \neq 0$

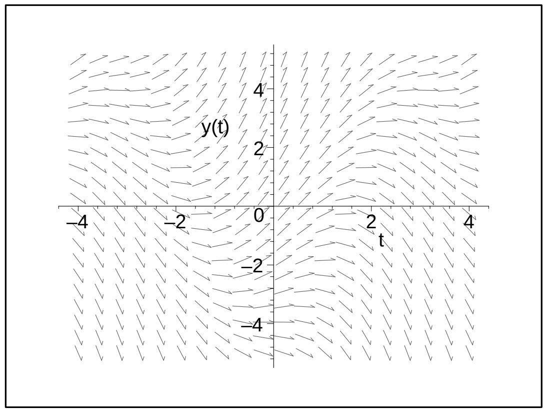
20. After dividing by t, we see that the integrating factor is $\mu(t) = te^t$. Therefore,

$$(te^ty)' = te^t \implies y = t^{-1}e^{-t} \int te^t\,dt = t^{-1}e^{-t}(te^t - e^t + c) = t^{-1}(t - 1 + ce^{-t}).$$

The initial condition $y(\ln 2) = 1$ implies $c = 2$ and $y = (t - 1 + 2e^{-t})/t, \quad t \neq 0$

21.

(a)



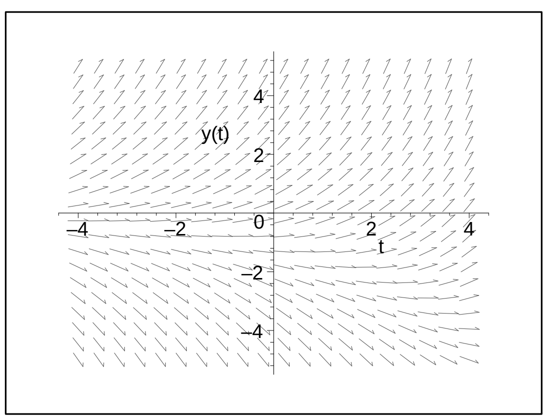
The solutions appear to diverge from an oscillatory solution. It appears that $a_0 \approx -1$. For $a > -1$, the solutions increase without bound. For $a < -1$, the solutions decrease without bound.

(b) The integrating factor is $\mu(t) = e^{-t/2}$. From this, we conclude that the general solution is $y(t) = (8\sin(t) - 4\cos(t))/5 + ce^{t/2}$. The solution will be sinusoidal as long as $c = 0$. The initial condition for the sinusoidal behavior is $y(0) = (8\sin(0) - 4\cos(0))/5 = -4/5$. Therefore, $a_0 = -4/5$.

(c) y oscillates for $a = a_0$

22.

(a)



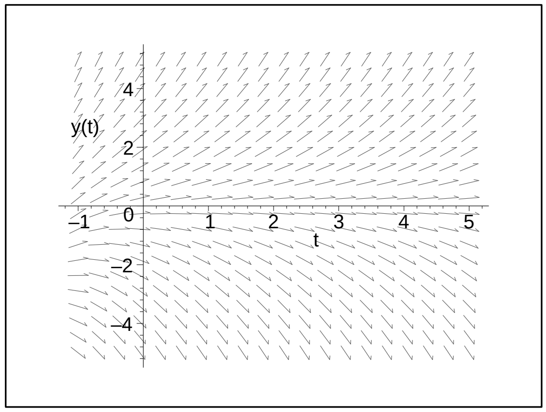
All solutions eventually increase or decrease without bound. The value a_0 appears to be approximately $a_0 = -3$.

(b) The integrating factor is $\mu(t) = e^{-t/2}$, and the general solution is $y(t) = -3e^{t/3} + ce^{t/2}$. The initial condition $y(0) = a$ implies $y = -3e^{t/3} + (a+3)e^{t/2}$. The solution will behave like $(a+3)e^{t/2}$. Therefore, $a_0 = -3$.

(c) $y \to -\infty$ for $a = a_0$

23.

(a)



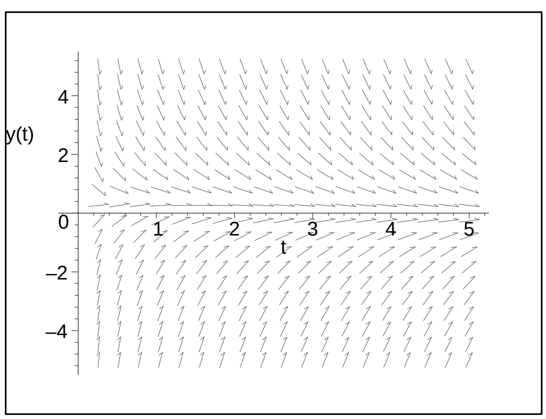
Solutions eventually increase or decrease without bound, depending on the initial value a_0. It appears that $a_0 \approx -1/8$.

(b) Dividing the equation by 3, we see that the integrating factor is $\mu(t) = e^{-2t/3}$. Therefore, the solution is $y = [(2 + a(3\pi + 4))e^{2t/3} - 2e^{-\pi t/2}]/(3\pi + 4)$. The solution will eventually behave like $(2 + a(3\pi + 4))e^{2t/3}/(3\pi + 4)$. Therefore, $a_0 = -2/(3\pi + 4)$.

(c) $y \to 0$ for $a = a_0$

24.

(a)



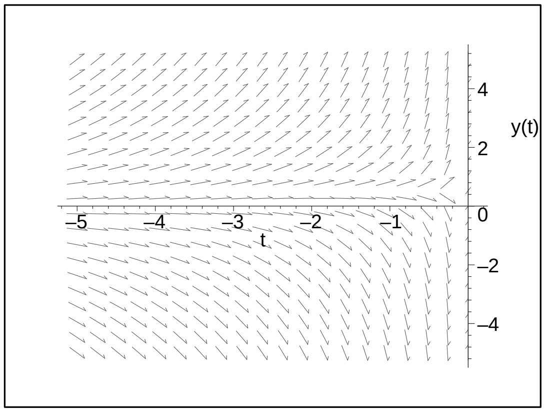
It appears that $a_0 \approx .4$. As $t \to 0$, solutions increase without bound if $y > a_0$ and decrease without bound if $y < a_0$.

(b) The integrating factor is $\mu(t) = te^t$. The general solution is $y = te^{-t} + ce^{-t}/t$. The initial condition $y(1) = a$ implies $y = te^{-t} + (ea - 1)e^{-t}/t$. As $t \to 0$, the solution will behave like $(ea - 1)e^{-t}/t$. From this, we see that $a_0 = 1/e$.

(c) $y \to 0$ as $t \to 0$ for $a = a_0$

25.

(a)



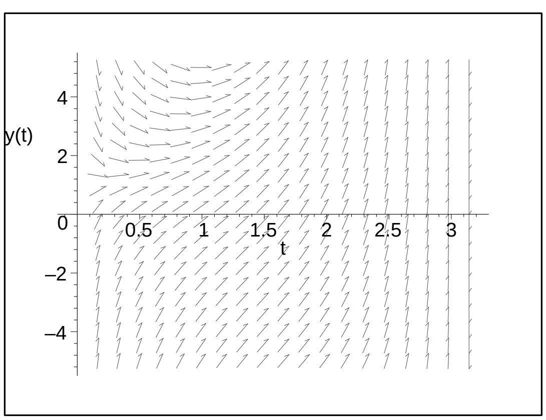
It appears that $a_0 \approx .4$. That is, as $t \to 0$, for $y(-\pi/2) > a_0$, solutions will increase without bound, while solutions will decrease without bound for $y(-\pi/2) < a_0$.

(b) After dividing by t, we see that the integrating factor is t^2, and the solution is $y = -\cos t/t^2 + \pi^2 a/4t^2$. Since $\lim_{t\to 0}\cos(t) = 1$, solutions will increase without bound if $a > 4/\pi^2$ and decrease without bound if $a < 4/\pi^2$. Therefore, $a_0 = 4/\pi^2$.

(c) For $a_0 = 4/\pi^2$, $y = (1 - \cos(t))/t^2 \to 1/2$ as $t \to 0$.

26.

(a)



It appears that $a_0 \approx 2$. For $y(1) > a_0$, the solution will increase without bound as $t \to 0$, while the solution will decrease without bound if $y(t) < a_0$.

(b) After dividing by $\sin(t)$, we see that the integrating factor is $\mu(t) = \sin(t)$. As a result, we see that the solution is given by $y = (e^t + c)\sin(t)$. Applying our initial condition, we see that our solution is $y = (e^t - e + a\sin 1)/\sin t$. The solution will increase if $1 - e + a\sin 1 > 0$ and decrease if $1 - e + a\sin 1 < 0$. Therefore, we conclude that $a_0 = (e-1)/\sin 1$

(c) If $a_0 = (e-1)\sin(1)$, then $y = (e^t - 1)/\sin(t)$. As $t \to 0$, $y \to 1$.

27. The integrating factor is $\mu(t) = e^{t/2}$. Therefore, the general solution is $y(t) = [4\cos(t) + 8\sin(t)]/5 + ce^{-t/2}$. Using our initial condition, we have $y(t) = [4\cos(t) + 8\sin(t) - 9e^{t/2}]/5$. Differentiating, we have

$$y' = [-4\sin(t) + 8\cos(t) + 4.5e^{-t/2}]/5$$
$$y'' = [-4\cos(t) - 8\sin(t) - 2.25e^{t/2}]/5.$$

Setting $y' = 0$, the first solution is $t_1 = 1.3643$, which gives the location of the first stationary point. Since $y''(t_1) < 0$, the first stationary point is a local maximum. The coordinates of the point are $(1.3643, .82008)$.

28. The integrating factor is $\mu(t) = e^{2t/3}$. The general solution of the differential equation is $y(t) = (21 - 6t)/8 + ce^{-2t/3}$. Using the initial condition, we have $y(t) = (21 - 6t)/8 + (y_0 - 21/8)e^{-2t/3}$. Therefore, $y'(t) = -3/4 - (2y_0 - 21/4)e^{-2t/3}/3$. Setting $y'(t) = 0$, the solution is $t_1 = \frac{3}{2}\ln[(21 - 8y_0)/9]$. Substituting into the solution, the respective value at the stationary point is $y(t_1) = \frac{3}{2} + \frac{9}{4}\ln 3 - \frac{9}{8}\ln(21 - 8y_0)$. Setting this result equal to zero, we obtain the required initial value $y_0 = (21 - 9e^{4/3})/8 = -1.643$.

29.

(a) The integrating factor is $\mu(t) = e^{t/4}$. The general solution is

$$y(t) = 12 + [8\cos(2t) + 64\sin(2t)]/65 + ce^{-t/4}.$$

Applying the initial condition $y(0) = 0$, we arrive at the specific solution

$$y(t) = 12 + [8\cos(2t) + 64\sin(2t) - 788e^{-t/4}]/65.$$

For large values of t, the solution oscillates about the line $y = 12$.

(b) To find the value of t for which the solution first intersects the line $y = 12$, we need to solve the equation $8\cos(2t) + 64\sin(2t) - 788e^{-t/4} = 0$. The time t is approximately 10.519.

30. The integrating factor is $\mu(t) = e^{-t}$. The general solution is $y(t) = -1 - \frac{3}{2}\cos(t) - \frac{3}{2}\sin(t) + ce^t$. In order for the solution to remain finite as $t \to \infty$, we need $c = 0$. Therefore, y_0 must satisfy $y_0 = -1 - 3/2 = -5/2$.

31. The integrating factor is $\mu(t) = e^{-3t/2}$ and the general solution of the equation is $y(t) = -2t-4/3-4e^t+ce^{3t/2}$. The initial condition implies $y(t) = -2t-4/3-4e^t+(y_0+16/3)e^{3t/2}$. The solution will behave like $(y_0+16/3)e^{3t/2}$ (for $y_0 \neq -16/3$). For $y_0 > -16/3$, the solutions will increase without bound, while for $y_0 < -16/3$, the solutions will decrease without bound. If $y_0 = -16/3$, the solution will decrease without bound as the solution will be $-2t-4/3-4e^t$.

32. By equation (41), we see that the general solution is given by

$$y = e^{-t^2/4}\int_0^t e^{s^2/4}\,ds + ce^{-t^2/4}.$$

Applying L'Hospital's rule,

$$\lim_{t\to\infty}\frac{\int_0^t e^{s^2/4}\,ds}{e^{t^2/4}} = \lim_{t\to\infty}\frac{e^{t^2/4}}{(t/2)e^{t^2/4}} = 0.$$

Therefore, $y \to 0$ as $t \to \infty$.

33. The integrating factor is $\mu(t) = e^{at}$. First consider the case $a \neq \lambda$. Multiplying the equation by e^{at}, we have

$$(e^{at}y)' = be^{(a-\lambda)t} \implies y = e^{-at}\int be^{(a-\lambda)t} = e^{-at}\left(\frac{b}{a-\lambda}e^{(a-\lambda)t} + c\right) = \frac{b}{a-\lambda}e^{-\lambda t} + ce^{-at}.$$

Since a, λ are assumed to be positive, we see that $y \to 0$ as $t \to \infty$. Now if $a = \lambda$ above, then we have

$$(e^{at}y)' = b \implies y = e^{-at}(bt+c)$$

and similarly $y \to 0$ as $t \to \infty$.

34. We notice that $y(t) = ce^{-t} + 3$ approaches 3 as $t \to \infty$. We just need to find a first-order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 3$. Then $f' + f = 0$ and $g' + g = 3$. Therefore, $y(t) = ce^{-t} + 3$ satisfies the equation $y' + y = 3$. That is, the equation $y' + y = 3$ has the desired properties.

35. We notice that $y(t) = ce^{-t} + 3 - t$ approaches $3 - t$ as $t \to \infty$. We just need to find a first-order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 3 - t$. Then $f' + f = 0$ and $g' + g = -1 + 3 - t = -2 - t$. Therefore, $y(t) = ce^{-t} + 3 - t$ satisfies the equation $y' + y = -2 - t$. That is, the equation $y' + y = -2 - t$ has the desired properties.

36. We notice that $y(t) = ce^{-t} + 2t - 5$ approaches $2t - 5$ as $t \to \infty$. We just need to find a first-order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 2t - 5$. Then $f' + f = 0$ and $g' + g = 2 + 2t - 5 = 2t - 3$. Therefore, $y(t) = ce^{-t} + 2t - 5-$ satisfies the equation $y' + y = 2t - 3$. That is, the equation $y' + y = 2t - 3$ has the desired properties.

37. We notice that $y(t) = ce^{-t} + 4 - t^2$ approaches $4 - t^2$ as $t \to \infty$. We just need to find a first-order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 4 - t^2$. Then $f' + f = 0$ and

$g' + g = -2t + 4 - t^2 = 4 - 2t - t^2$. Therefore, $y(t) = ce^{-t} + 2t - 5-$ satisfies the equation $y' + y = 4 - 2t - t^2$. That is, the equation $y' + y = 4 - 2t - t^2$ has the desired properties.

38. Multiplying the equation by $e^{a(t-t_0)}$, we have

$$\begin{aligned} &e^{a(t-t_0)}y' + ae^{a(t-t_0)}y = e^{a(t-t_0)}g(t) \\ \implies &(e^{a(t-t_0)}y)' = e^{a(t-t_0)}g(t) \\ \implies &y(t) = \int_{t_0}^{t} e^{-a(t-s)}g(s)\,ds + e^{-a(t-t_0)}y_0. \end{aligned}$$

Assuming $g(t) \to g_0$ as $t \to \infty$,

$$\int_{t_0}^{t} e^{-a(t-s)}g(s)\,ds \to \int_{t_0}^{t} e^{-a(t-s)}g_0\,ds = \frac{g_0}{a} - \frac{e^{-a(t-t_0)}}{a}g_0 \to \frac{g_0}{a} \quad \text{as } t \to \infty$$

For an example, let $g(t) = e^{-t} + 1$. Assume $a \neq 1$. By undetermined coefficients, we look for a solution of the form $y = ce^{-at} + Ae^{-t} + B$. Substituting a function of this form into the differential equation leads to the equation

$$[-A + aA]e^{-t} + aB = e^{-t} + 1 \implies -A + aA = 1 \text{ and } aB = 1.$$

Therefore, $A = 1/(a-1)$, $B = 1/a$ and $y = ce^{-at} + \frac{1}{a-1}e^{-t} + 1/a$. The initial condition $y(0) = y_0$ implies $y(t) = (y_0 - \frac{1}{a-1} - \frac{1}{a})e^{-at} + \frac{1}{a-1}e^{-t} + 1/a \to 1/a$ as $t \to \infty$.

39.

(a) The integrating factor is $e^{\int p(t)\,dt}$. Multiplying by the integrating factor, we have

$$e^{\int p(t)\,dt}y' + e^{\int p(t)\,dt}p(t)y = 0.$$

Therefore,

$$\left(e^{\int p(t)\,dt}y\right)' = 0$$

which implies

$$y(t) = Ae^{-\int p(t)\,dt}$$

is the general solution.

(b) Let $y = A(t)e^{-\int p(t)\,dt}$. Then in order for y to satisfy the desired equation, we need

$$A'(t)e^{-\int p(t)\,dt} - A(t)p(t)e^{-\int p(t)\,dt} + A(t)p(t)e^{-\int p(t)\,dt} = g(t).$$

That is, we need

$$A'(t) = g(t)e^{\int p(t)\,dt}.$$

(c) From equation (iv), we see that

$$A(t) = \int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C.$$

Therefore,

$$y(t) = e^{-\int p(t)\,dt}\left(\int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C\right).$$