Section 1.4

1. The differential equation is second order, since the highest derivative in the equation is of order two. The equation is linear since the left hand side is a linear function of y and its derivatives and the right hand side is just a function of t.

2. The differential equation is second order since the highest derivative in the equation is of order two. The equation is nonlinear because of the term $y^2 d^2y/dt^2$.

3. The differential equation is fourth order since the highest derivative in the equation is of order four. The equation is linear since the left hand side is a linear function of y and its derivatives and the right hand side does not depend on y.

4. The differential equation is first order since the only derivative in the equation is of order one. The equation is nonlinear because of the y^2 term.

5. The differential equation is second order since the highest derivative in the equation is of order two. The equation is nonlinear because of the term $\sin(t+y)$ which is not a linear function of y.

6. The differential equation is third order since the highest derivative in the equation is of order three. The equation is linear because the left hand side is a linear function of y and its derivatives, and the right hand side is only a function of t.

7. $y_1(t) = e^t \implies y_1' = e^t \implies y_1'' = e^t$. Therefore, $y_1'' - y_1 = 0$. Also, $y_2 = \cosh t \implies y_2' = \sinh t \implies y_2' = \cosh t$. Therefore, $y_2'' - y_2 = 0$.

8. $y_1 = e^{-3t} \implies y_1' = -3e^{-3t} \implies y_1'' = 9e^{-3t}$. Therefore, $y_1'' + 2y_1' - 3y_1 = (9-6-3)y_1 = 0$. Also, $y_2 = e^t \implies y_2' = y_2'' = e^t$. Therefore, $y_1'' + 2y_1' - 3y_1 = (1+2-3)e^t = 0$.

9. $y = 3t + t^2 \implies y' = 3 + 2t$. Therefore, $ty' - y = t(3+2t) - (3t + t^2) = t^2$.

10. $y_1 = t/3 \implies y_1' = 1/3 \implies y_1'' = y_1''' = y_1'''' = 0$. Therefore, $y_1'''' + 4y_1''' + 3y = t$. Also, $y_2 = e^{-t} + t/3 \implies y_2' = -e^{-t} + 1/3 \implies y_2'' = e^{-t} \implies y_2''' = -e^{-t} \implies y_2'''' = e^{-t}$. Therefore, $y_2'''' + 4y_2''' + 3y = e^{-t} - 4e^{-t} + 3(e^{-t} + t/3) = t$.

11. $y_1 = t^{1/2} \implies y_1' = t^{-1/2}/2 \implies y_1'' = -t^{-3/2}/4$. Therefore, $2t^2y_1'' + 3ty_1' - y_1 = 2t^2(-t^{-3/2}/4) + 3t(t^{-1/2}/2) - t^{1/2} = (-1/2 + 3/2 - 1)t^{1/2} = 0$. Also, $y_2 = t^{-1} \implies y_2' = -t^{-2} \implies y_2'' = 2t^{-3}$. Therefore, $2t^2y_2'' + 3ty_2' - y_2 = 2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} = (4 - 3 - 1)t^{-1} = 0$.

12. $y_1 = t^{-2} \implies y_1' = -2t^{-3} \implies y_1'' = 6t^{-4}$. Therefore, $t^2y_1'' + 5ty_1' + 4y_1 = t^2(6t^{-4}) + 5t(-2t^{-3}) + 4t^{-2} = (6-10+4)t^{-2} = 0$. Also, $y_2 = t^{-2}\ln t \implies y_2' = t^{-3} - 2t^{-3}\ln t \implies y_2'' = -5t^{-4} + 6t^{-4}\ln t$. Therefore, $t^2y_2'' + 5ty_2' + 4y_2 = t^2(-5t^{-4} + 6t^{-4}\ln t) + 5t(t^{-3} - 2t^{-3}\ln t) + 4(t^{-2}\ln t) = (-5+5)t^{-2} + (6-10+4)t^{-2}\ln t = 0$.

13. $y = (\cos t)\ln\cos t + t\sin t \implies y' = -(\sin t)\ln\cos t + t\cos t \implies y'' = -(\cos t)\ln\cos t - t\sin t + \sec t$. Therefore, $y'' + y = -(\cos t)\ln\cos t - t\sin t + \sec t + (\cos t)\ln\cos t + t\sin t = \sec t$.

14. $y = e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2} \implies y' = 2te^{t^2}\int_0^t e^{-s^2}\,ds + 1 + 2te^{t^2}$. Therefore, $y' - 2ty = 2te^{t^2}\int_0^t e^{-s^2}\,ds + 1 + 2te^{t^2} - 2t(e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}) = 1$.

15. Let $y = e^{rt}$. Then $y' = re^{rt}$. Substituting these terms into the differential equation, we have $y' + 2y = re^{rt} + 2e^{rt} = (r+2)e^{rt} = 0$. This equation implies $r = -2$.

16. Let $y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Substituting these terms into the differential equation, we have $y'' - y = (r^2 - 1)e^{rt} = 0$. This equation implies $r = \pm 1$.

17. Let $y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Substituting these terms into the differential equation, we have $y'' + y' - 6y = (r^2 + r - 6)e^{rt} = 0$. In order for r to satisfy this equation, we need $r^2 + r - 6 = 0$. That is, we need $r = 2, -3$.

18. Let $y = e^{rt}$. Then $y' = re^{rt}$, $y'' = r^2e^{rt}$ and $y''' = r^3e^{rt}$. Substituting these terms into the differential equation, we have $y''' - 3y'' + 2y' = (r^3 - 3r^2 + 2r)e^{rt} = 0$. In order for r to satisfy this equation, we need $r^3 - 3r^2 + 2r = 0$. That is, we need $r = 0, 1, 2$.

19. Let $y = t^r$. Then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these terms into the differential equation, we have $t^2y'' + 4ty' + 2y = t^2(r(r-1)t^{r-2}) + 4t(rt^{r-1}) + 2t^r = (r(r-1) + 4r + 2)t^r = 0$. In order for r to satisfy this equation, we need $r(r-1) + 4r + 2 = 0$. Simplifying this expression, we need $r^2 + 3r + 2 = 0$. The solutions of this equation are $r = -1, -2$.

20. Let $y = t^r$. Then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these terms into the differential equation, we have $t^2y'' - 4ty' + 4y = t^2(r(r-1)t^{r-2}) - 4t(rt^{r-1}) + 4t^r = (r(r-1) - 4r + 4)t^r = 0$. In order for r to satisfy this equation, we need $r(r-1) - 4r + 4 = 0$. Simplifying this expression, we need $r^2 - 5r + 4 = 0$. The solutions of this equation are $r = 1, 4$.

21.

(a) Consider Figure 1.4.1 in the text. There are two main forces acting on the mass: (1) the tension in the rod and (2) gravity. The tension, T, acts on the mass along the direction of the rod. By extending a line below and to the right of the mass at an angle θ with the vertical, we see that there is a force of magnitude $mg\cos\theta$ acting on the mass in that direction. Then extending a line below the mass and to the left, making an angle of $\pi - \theta$, we see the force acting on the mass in the tangential direction is $mg\sin\theta$.

(b) Newton's Second Law states that $\sum \mathbf{F} = m\mathbf{a}$. In the tangential direction, the equation of motion may be expressed as $\sum F_\theta = ma_\theta$. The tangential acceleration, a_θ is the linear acceleration along the path. That is, $a_\theta = Ld^2\theta/dt^2$. The only force acting in the tangential direction is the gravitational force in the tangential direction which is given by $-mg\sin\theta$. Therefore, $-mg\sin\theta = mLd^2\theta/dt^2$.

(c) Rearranging terms, we have

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta.$$

22.

(a) The kinetic energy of a particle of mass m is given by $T = \frac{1}{2}mv^2$ where v is its speed. A particle in motion on a circle of radius L has speed $L(d\theta/dt)$ where $d\theta/dt$ is its angular speed. Therefore,

$$T = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2.$$

(b) The potential energy of a particle is given by $V = mgh$ where h is the height above some point. Here we measure h as the height above the pendulum's lowest position. Since $\dfrac{L-h}{L} = \cos\theta$, we have $h = L(1-\cos\theta)$. Therefore,

$$V = mgL(1-\cos\theta).$$

(c) Since the total energy is conserved. We know that for $E = T + V$, $dE/dt = 0$. Here,

$$E = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1-\cos\theta)$$

implies

$$\frac{dE}{dt} = mL^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgL\sin\theta\frac{d\theta}{dt} = 0.$$

Simplifying this equation, we conclude that

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

23.

(a) Angular momentum is the moment about a certain point of linear momentum, which is given by

$$mv = mL\frac{d\theta}{dt}.$$

The moment about a pivot point is given by

$$M_p = mL^2\frac{d\theta}{dt}.$$

(b) The moment of the gravitational force is

$$M_g = -mg \cdot L\sin\theta.$$

Then $dM_p/dt = M_g$ implies

$$mL^2\frac{d^2\theta}{dt^2} = -mgL\sin\theta.$$

Rewriting this equation, we have

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$