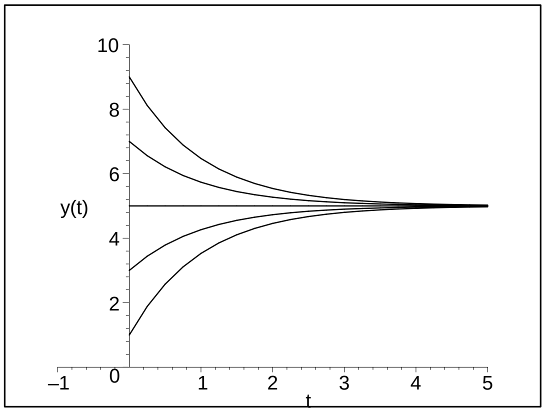
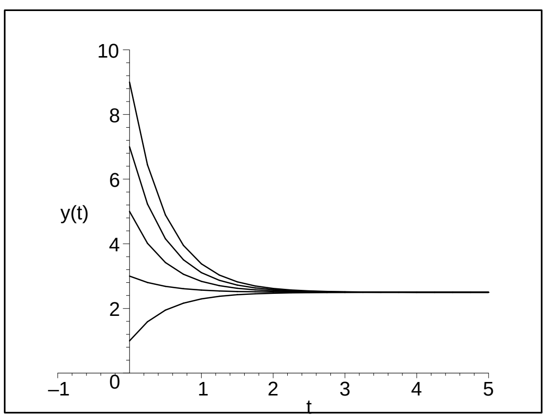
1.

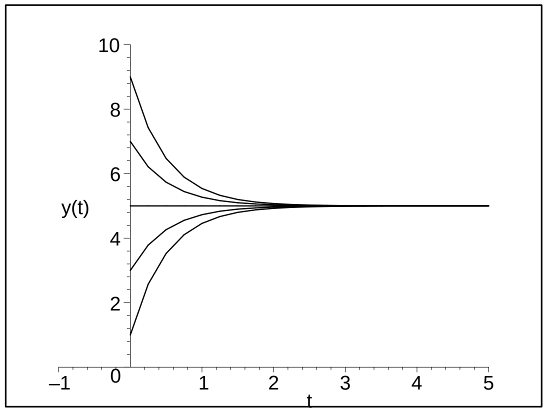
(a) Rewrite the equation as
$$\frac{dy}{5-y} = dt$$
and then integrate both sides. Doing so, we see that $-\ln|5-y| = t + c$. Applying the exponential function, we have $5 - y = ce^{-t}$. Substituting in our initial condition $y(0) = y_0$, we have $5 - y_0 = c$. Therefore, our solution is $y(t) = 5 + (y_0 - 5)e^{-t}$.



(b) Rewrite the equation as
$$\frac{dy}{5-2y} = dt$$
and then integrate both sides. Doing so, we see that $\ln|5-2y| = -2t + c$. Applying the exponential function, we have $5 - 2y = ce^{-2t}$. Substituting in our initial condition $y(0) = y_0$, we have $5 - 2y_0 = c$. Therefore, our solution is $y(t) = (5/2) + [y_0 - (5/2)]e^{-2t}$



(c) Rewrite the equation as
$$\frac{dy}{10-2y} = dt$$
and then integrate both sides. Doing so, we see that $\ln|10-2y| = -2t + c$. Applying the exponential function, we have $10 - 2y = ce^{-2t}$. Substituting in our initial condition $y(0) = y_0$, we have $10 - 2y_0 = c$. Therefore, our solution is $y(t) = 5 + [y_0 - 5]e^{-2t}$



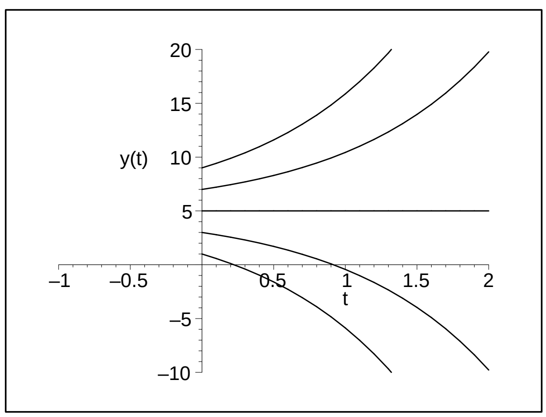
The equilibrium solution is $y = 5$ in (a) and (c), $y = 5/2$ in (b). The solution approaches equilibrium faster in (b) and (c) than in (a).

2.

(a) Rewrite the equation as

$$\frac{dy}{y-5} = dt$$

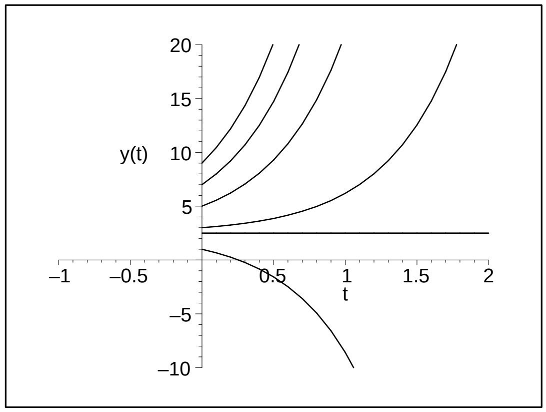
and then integrate both sides. Doing so, we see that $\ln|y-5| = t + c$. Applying the exponential function, we have $y - 5 = ce^t$. Substituting in our initial condition $y(0) = y_0$, we have $y_0 - 5 = c$. Therefore, our solution is $y(t) = 5 + [y_0 - 5]e^t$



(b) Rewrite the equation as

$$\frac{dy}{2y-5} = dt$$

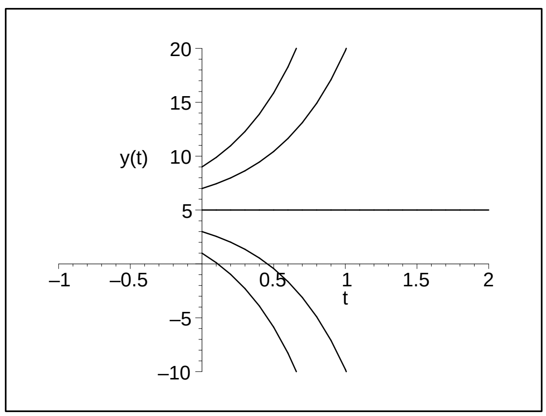
and then integrate both sides. Doing so, we see that $\ln|2y-5| = 2t + c$. Applying the exponential function, we have $2y - 5 = ce^{2t}$. Substituting in our initial condition $y(0) = y_0$, we have $2y_0 - 5 = c$. Therefore, our solution is $y(t) = (5/2) + [y_0 - (5/2)]e^{2t}$



(c) Rewrite the equation as

$$\frac{dy}{2y-10} = dt$$

and then integrate both sides. Doing so, we see that $\ln|2y - 10| = 2t + c$. Applying the exponential function, we have $2y - 10 = ce^{2t}$. Substituting in our initial condition $y(0) = y_0$, we have $2y_0 - 10 = c$. Therefore, our solution is $y(t) = 5 + [y_0 - 5]e^{2t}$



The equilibrium solution is $y = 5$ in (a) and (c), $y = 5/2$ in (b); solution diverges from equilibrium faster in (b) and (c) than in (a).

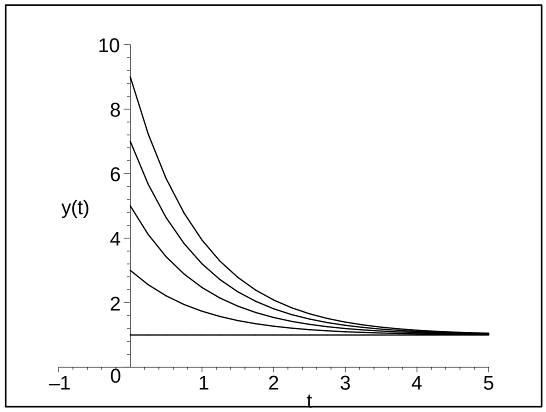
3.

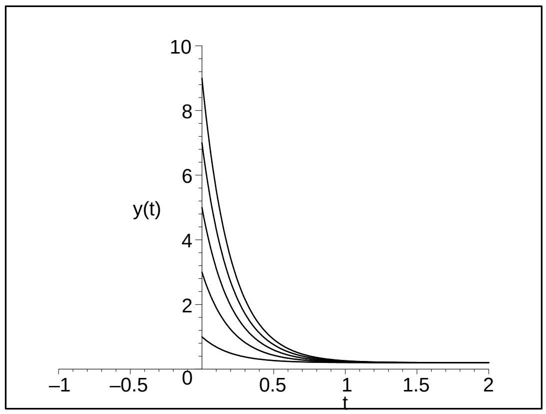
(a) Rewrite the equation as

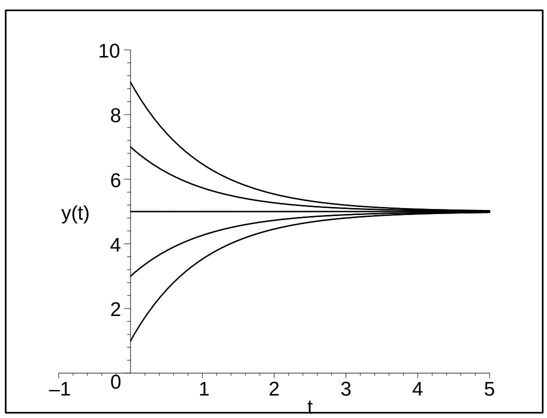
$$\frac{dy}{b-ay} = dt$$

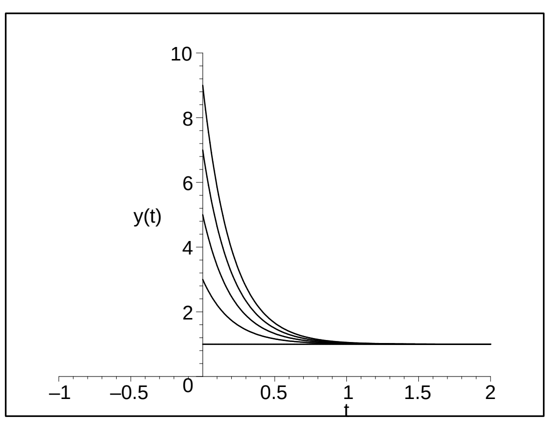
and then integrate both sides. Doing so, we see that $\ln|b - ay| = -at + c$. Applying the exponential function, we have $b - ay = ce^{-at}$, or $y = ce^{-at} + (b/a)$

(b) Below we show solution curves for various initial conditions under the cases $a = 1, b = 1$, $a = 5, b = 1$, $a = 1, b = 5$ and $a = 5, b = 5$, respectively.


y(t)
t


y(t)
t


y(t)
t


y(t)
t

(c) (i) As a increases, the equilibrium is lower and is approached more rapidly. (ii) As b increases, the equilibrium is higher. (iii) As a and b increase, but a/b remains the same, the equilibrium remains the same and is approached more rapidly.

4.

(a) The equilibrium solution occurs when $dy/dt = ay - b = 0$. Therefore, the equilibrium solution is $y_e = b/a$

(b) If $Y(t) = y - y_e$, then $Y'(t) = y' - y_e' = y' = ay - b = a(Y + y_e) - b = aY + ay_e - b = aY + a(b/a) - b = 0 = aY$. Therefore, Y satisfies the equation $Y' = aY$.

5. The solution of the homogeneous problem is $y = ce^{-2t}$. Therefore, we assume the solution will have the form $y = ce^{-2t} + At + B$. Substituting a function of this form into the differential equation leads to the equation

$$2At + A + 2B = t - 3.$$

Equating like coefficients, we see that $A = 1/2$ and $B = -7/4$. Therefore, the general solution is

$$y = ce^{-2t} + \frac{1}{2}t - \frac{7}{4}.$$

6. The solution of the homogeneous problem is $y = ce^{3t}$. Therefore, we assume the solution will have the form $y = ce^{3t} + Ae^{-t}$. Substituting a function of this form into the differential equation leads to the equation

$$-4Ae^{-t} = e^{-t}.$$

Equating like coefficients, we see that $A = -1/4$. Therefore, the general solution is

$$y = ce^{3t} - \frac{1}{4}e^{-t}.$$

7. The solution of the homogeneous problem is $y = ce^{-t}$. Therefore, we assume the solution will have the form $y = ce^{-t} + A\cos(2t) + B\sin(2t)$. Substituting a function of this form into the differential equation leads to the equation

$$[-2A + B]\sin(2t) + [2B + A]\cos(2t) = 3\cos(2t).$$

Solving the two equations, $-2A + B = 0$ and $2B + A = 3$, we see that $A = 3/5$ and $B = 6/5$. Therefore, the general solution is

$$y = ce^{-t} + \frac{3}{5}\cos(2t) + \frac{6}{5}\sin(2t).$$

8. The solution of the homogeneous problem is $y = ce^{2t}$. Therefore, we assume the solution will have the form $y = ce^{2t} + A\cos(t) + B\sin(t)$. Substituting a function of this form into the differential equation leads to the equation

$$[-A - 2B]\sin(t) + [B - 2A]\cos(t) = 2\sin(t).$$

Solving the system of equations $-A - 2B = 2$ and $B - 2A = 0$, we see that $A = -2/5$ and $B = -4/5$. Therefore, the general solution is

$$y = ce^{2t} - \frac{2}{5}\cos(t) - \frac{4}{5}\sin(t).$$

9. The solution of the homogeneous problem is $y = ce^{-2t}$. Therefore, we assume the solution will have the form $y = ce^{-2t} + At + B + C\cos(t) - D\sin(t)$. Substituting a function of this form into the differential equation leads to the equation

$$2At + [A + 2B] + [C + 2D]\cos(t) + [2C - D]\sin(t) = 2t + 3\sin(t).$$

Equating like coefficients, we see that $A = 1$, $B = -1/2$, $C = 6/5$ and $D = -3/5$. Therefore, the general solution is

$$y = ce^{-2t} + t - \frac{1}{2} + \frac{6}{5}\sin(t) - \frac{3}{5}\cos(t).$$

10. The solution of the homogeneous problem is $y = ce^{2t}$. Therefore, we assume the solution will have the form $y = ce^{2t} + Ae^t + Bt^2 + Ct + D$. Substituting a function of this form into the differential equation leads to the equation

$$-Ae^t - 2Bt^2 + [2B - 2C]t + [C - 2D] = 3e^t + t^2 + 1.$$

Equating like coefficients, we see that $A = -3$, $B = -1/2$, $C = -1/2$ and $D = -3/4$. Therefore, the general solution is

$$y = ce^{2t} - 3e^t - \frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}.$$

11.

(a) The general solution is $p(t) = 900 + ce^{t/2}$. Plugging in for the initial condition, we have $p(t) = 900 + (p_0 - 900)e^{t/2}$. With $p_0 = 850$, the solution is $p(t) = 900 - 50e^{t/2}$. To find the time when the population becomes extinct, we need to find the time T when $p(T) = 0$. Therefore, $900 = 50e^{T/2}$, which implies $e^{T/2} = 18$, and, therefore, $T = 2\ln 18 \cong 5.78$ months.

(b) Using the general solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, we see that the population will become extinct at the time T when $900 = (900 - p_0)e^{T/2}$. That is, $T = 2\ln[900/(900 - p_0)]$ months

(c) Using the general solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, we see that the population after 1 year (12 months) will be $p(6) = 900 + (p_0 - 900)e^6$. If we want to know the initial population which will lead to extinction after 1 year, we set $p(6) = 0$ and solve for p_0. Doing so, we have $(900 - p_0)e^6 = 900$ which implies $p_0 = 900(1 - e^{-6}) \cong 897.8$

12.

(a) The general solution is $p(t) = p_0 e^{rt}$, where t is measured in days. If the population doubles in 30 days, then $p(30) = 2p_0 = p_0 e^{30r}$. Therefore, $r = (\ln 2)/30$ day^{-1}.

(b) If the population doubles in N days, then $p(N) = 2p_0 = p_0 e^{Nr}$. Therefore, $r = (\ln 2)/N$ day^{-1}

13.

(a) The solution is given by $v(t) = 35(1 - e^{-0.28t})$. The limiting velocity is 35 m/sec. Therefore, we want to find the time T when $v(T) = .98 \cdot 35 = 34.3$ m/sec. Plugging this value into our equation for v, we have $34.3 = 35(1 - e^{-0.28T})$, or $e^{-0.28T} = .02$ which implies $T = (\ln 50)/0.28 \cong 13.97$ sec

(b) To find the position, we integrate the velocity function above. For $v(t) = 35(1 - e^{-0.28t})$, the height is given by $s(t) = \int v(t) = 35t + 125te^{-0.28t}\,dt + C$. Assuming, $s(0) = 0$, we see that $c = -125$. Therefore, $s(t) = 35t + 125e^{-0.28t} - 125$. When $T = 13.97$ seconds, we see that the distance traveled is approximately 366.5 m.

14.

(a) Assuming no air resistance, Newton's Second Law can be expressed as

$$m\frac{dv}{dt} = mg$$

where g is the gravitational constant. Dividing the above equation by m and assuming that the initial velocity is zero, we see that our initial value problem is $dv/dt = 9.8$, $v(0) = 0$

(b) We are assuming the object is released from a height of 300 meters above the ground. The height at a later time t satisfies $ds/dt = v = 9.8t$. Taking the point of release as the origin and integrating the above equation for s, we have $s(t) = 4.9t^2$. We need to find the time T when $s(T) = 300$. That is, $4.9T^2 = 300$. Solving this equation, we have $T = \sqrt{300/4.9} \cong 7.82$ sec

(c) Using the equation $v = 9.8t$, we see that when $T \cong 7.82$ seconds, $v \cong 76.68$ m/sec

15.

(a) If we are assuming that the drag force is proportional to the square of the velocity, equation (1) becomes

$$m\frac{dv}{dt} = mg - \gamma v^2.$$

Plugging in $m = 0.025$, $g = 9.8$, the equation can be written as

$$\frac{dv}{dt} = 9.8 - \frac{\gamma}{.025}v^2.$$

If the limiting velocity is 35 m/sec, then $\gamma(35)^2 = 9.8 \cdot .025$ which implies $\gamma = 0.0002$. Therefore,

$$\frac{dv}{dt} = 9.8 - 0.008v^2,$$

or

$$\frac{dv}{dt} = [(35)^2 - v^2]/125.$$

(b) The equation can be rewritten as

$$\frac{dv}{(35)^2 - v^2} = \frac{dt}{125}.$$

Integrating both sides, we have

$$\ln\left|\frac{v+35}{v-35}\right| = \frac{70}{125}t + c.$$

Plugging in the initial condition $v(0) = 0$, we have $c = 0$. Applying the exponential function to both sides of the equation, we have

$$v + 35 = e^{70t/125}(35 - v).$$

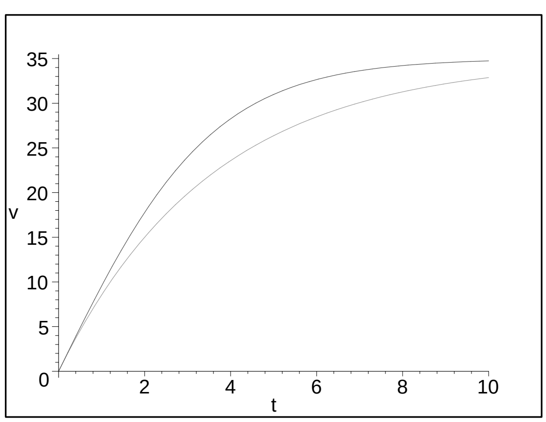
Solving this equation for v, we have

$$v(t) = 35\left[\frac{e^{70t/125} - 1}{e^{70t/125} + 1}\right]$$

or

$$v(t) = 35\left[\frac{e^{35t/125}(e^{35t/125} - e^{-35t/125})}{e^{35t/125}(e^{35t/125} + e^{-35t/125})}\right] = 35\tanh(7t/25)$$

(c) Below we show the graphs of $v(t)$ above (the top curve) and the solution to the problem in example 2 (the bottom curve)



(d) The quadratic force leads to the falling object attaining its limiting velocity sooner.

(e) The distance $x(t) = \int v(t)\,dt = \int 35\tanh(7t/25)\,dt = 125\ln\cosh(7t/25)$.

(f) Plugging 300 in for $x(T)$ in the answer to part (d), we have $300 = 125\ln\cosh(7T/25)$. Therefore, $T = (25/7)\text{arccosh}(e^{12/5}) \cong 11.04$ sec

16.

(a) The general solution of the equation is $Q(t) = ce^{-rt}$. Given that $Q(0) = 100$, we have $c = 100$. Assuming that $Q(1) = 82.04$, we have $82.04 = 100e^{-r}$. Solving this equation for r, we have $r = -\ln(82.04/100) = .19796$ per week or $r = 0.02828$ per day.

(b) Using the form of the general solution and r found above, we have $Q(t) = 100e^{-0.02828t}$

(c) Let T be the time it takes the isotope to decay to half of its original amount. From part (b), we conclude that $.5 = e^{-0.2828T}$ which implies that $T = -\ln(0.5)/0.2828 \cong 24.5$ days

17. The general solution of the differential equation is $Q(t) = Q_0e^{-rt}$ where $Q_0 = Q(0)$. Let τ be the half-life. Plugging τ into the equation for Q, we have $0.5Q_0 = Q_0e^{-r\tau}$. Therefore, $0.5 = e^{-r\tau}$ which implies $\tau = -\ln(0.5)/r = \ln(2)/r$. Therefore, we conclude that $r\tau = \ln 2$.

18. The differential equation for radium-226 is $dQ/dt = -rQ$. The solution of this equation is $Q(t) = Q_0e^{-rt}$. Using the result from exercise 17 and the fact that the half-life is 1620 years, we conclude that the decay rate $r = \ln(2)/\tau = \ln(2)/1620$. Let T be the time it takes for the isotope to decay to 3/4 of its original amount. Then

$$\frac{3}{4}Q_0 = Q_0e^{-\ln(2)T/1620}$$

which implies $T = -1620\ln(3/4)/\ln(2) \cong 672.4$ years.

19.

(a) We rewrite the equation as

$$\frac{du}{u-T} = -k.$$

Integrating both sides, we have $\ln|u-T| = -kt + c$. Applying the exponential function to both sides of the equation and plugging in the initial condition $u(0) = u_0$, we arrive at the general solution $u(t) = T + (u_0 - T)e^{-kt}$

(b) Since T is a constant, we see that if u satisfies the equation $du/dt = -k(u-T)$, then $d(u-T)/dt = du/dt = -k(u-T)$. Then using the result from exercise 17 above, we know that the relationship between the decay rate k and the time τ when the temperature difference is reduced by half satisfies the relationship $k\tau = \ln 2$.

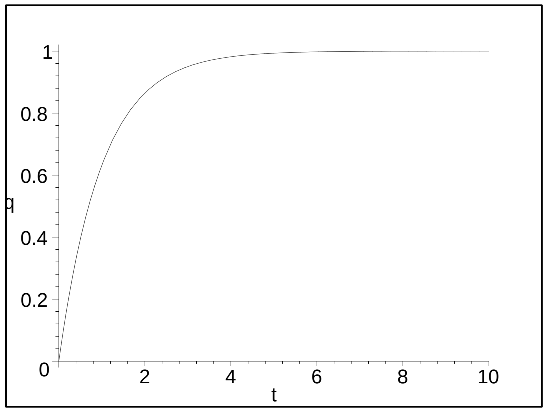
20. Based on exercise 19 above, the differential equation for the temperature in the room is given by

$$\frac{du}{dt} = -.15(u-10)$$

with an initial condition of $u(0) = 70$. As shown in exercise 19 above, the solution is given by $u(t) = 10 + 60e^{-0.15t}$. We need to find the time t such that $u(t) = 32$. That is, $22 = 60e^{-0.15t}$. Solving this equation for t, we have $t = -\ln(22/60)/0.15 \cong 6.69$ hours.

21.

(a) The solution of the differential equation with $q(0) = 0$ is $q(t) = CV(1 - e^{-t/RC})$. Below we show a sketch in the case when $C = V = R = 1$.

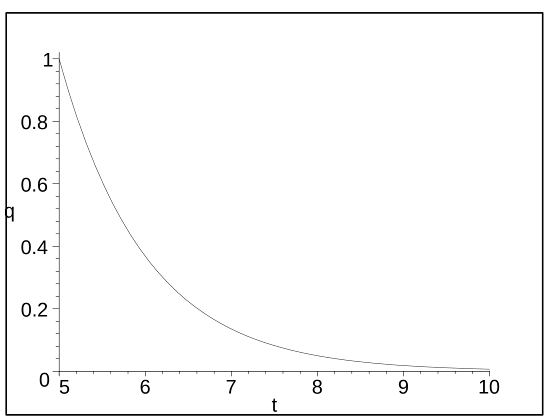


(b) As $t \to \infty$, the exponential term vanishes. Therefore, the limiting value is $q_L = CV$

(c) If the battery is removed, then $V = 0$. Therefore, our differential equation is

$$R\frac{dq}{dt} + \frac{q}{C} = 0.$$

Also, we are assuming that $q(t_1) = q_L = CV$. Solving the differential equation, we have $q = ce^{-t/RC}$. Using the initial condition $q(t_1) = CV$, we have $q(t_1) = ce^{-t_1/RC} = CV$. Therefore, $c = CVe^{t_1/RC}$. We conclude that $q(t) = CV\exp[-(t - t_1)/RC]$ Below we show a graph of the solution taking $C = V = R = 1$ and $t_1 = 5$.



22.

(a) The accumulation rate of the chemical is (0.01)(300) grams per hour. At any given time t, the concentration of the chemical in the pond is $Q(t)/10^6$ grams per gallon. Therefore, the chemical leaves the pond at the rate of $300Q(t)/10^6$ grams per hour. Therefore, the equation for Q is given by $Q' = 3(1 - 10^{-4}Q)$. Since initially there are no chemicals in the pond, $Q(0) = 0$.

(b) Rewrite the equation as

$$\frac{dQ}{10000 - Q} = 0.0003dt.$$

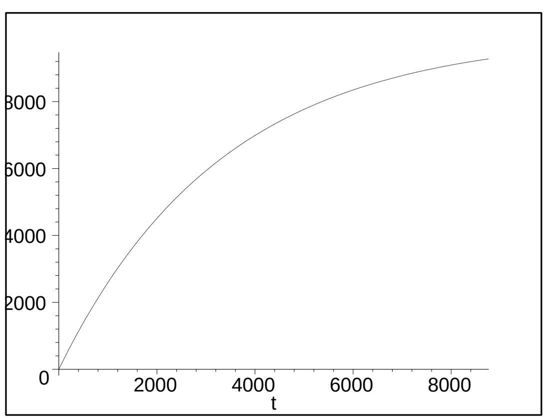
Integrating both sides of this equation, we have $\ln|10000 - Q| = -0.0003t + C$. Applying the exponential function to both sides of this equation, we have $10000 - Q = ce^{-0.0003t}$. Assuming $Q(0) = 0$, we see that $c = 10000$. Therefore, $Q(t) = 10000(1 - e^{-0.0003t})$ where t is measured in hours. Since 1 year is 8760 hours, we see that the amount of chemical in the pond after 1 year is $Q(8760) = 10000(1 - e^{-0.0003t}) \cong 9277.77$ grams.

(c) With the accumulation rate now equal to zero, the equation becomes $dQ/dt = -0.0003Q(t)$ grams/hour. Resetting the time variable, we assign the new initial value as $Q(0) =$ 9277.77 grams.

(d) The solution of the differential equation is $Q(t) = 9277.77e^{-0.0003t}$ after t hrs. Therefore, after 1 year $Q(8760) \cong 670.07$ g

(e) Letting T be the amount of time after the source is removed, we obtain the equation $10 = 9277.77e^{-0.0003t}$. Solving this equation, we have $T - \ln(10/9277.77)/0.0003 \cong 2.60$ years

(f)



23.

(a) We are assuming that no dye is entering the pool. The rate at which the dye is leaving the pool is given by $200 \cdot (q/60,000)$ g/min $= q/300$ g/min. Since initially, there are 5 kg of the dye in the pool, the initial value problem is $q' = -q/300$, $q(0) = 5000$ g

(b) The solution of this initial value problem is $q(t) = 5000e^{-t/300}$ where g is in grams and t is in minutes.

(c) In 4 hours (240 minutes), the amount of dye in the pool will be $q(240) \cong 2246.6$ grams. Since there is $60,000$ gallons of water in the pool, the concentration will be $2246.6/60,000 \cong 0.0374$ grams/gallon. So, no, the pool will not be reduced to the desired level within 4 hours.

(d) Let T be the time that it takes to reduce the concentration level of the dye to 0.02 grams/gallon. At that time, the amount of dye in the pool needs to be 1200 grams (as $1200/60000 = 0.02$). Plugging $q(T) = 1200$ into our equation for q, we have $1200 = 5000e^{-T/300}$. Solving this equation, we have $T = 300 \ln(25/6) \cong 7.136$ hr

(e) Let r be the necessary flow rate. As in part (a), if the water leaves the pool at the rate of r gallons/minute, then the initial value problem will be $q' = -rq/60,000$, $q(0) = 5000$. The solution of this initial value problem is given by $q(t) = 5000e^{-rt/60,000}$. We need to fid the decay rate r such that when $t = 240$ minutes, the amount of dye $q = 1200$ grams. That is, we need to solve the equation $1200 = 5000e^{-240r/60,000}$. Solving this equation, we have $r = 250 \ln(25/6) \cong 256.78$ gal/min